THE SUMMABILITY OF THE DEVELOPMENTS IN BESSEL FUNCTIONS, WITH APPLICATIONS*

BY

CHARLES N. MOORE

Introduction.

FEJÉR has shown† that the Fourier development corresponding to a function $f(x)$, of period 2π, will be summable‡ to the value

$$\tfrac{1}{2}[f(x+0)+f(x-0)]$$

at all points at which the function $f(x)$ is continuous or has a finite jump, if $f(x)$ is finite and integrable or even if $f(x)$ becomes infinite at a finite number of points, provided it remains integrable. He has further shown in the article referred to that the development is uniformly summable throughout any closed interval which does not include a point of discontinuity of the function. He has used these properties in connection with a general theorem about convergence factors to establish the fact that the formal results obtained in the discussion of many problems in Mathematical Physics which involve the development of an arbitrary function in a Fourier's series really furnish a solution of the problem, even when the development of the function is a divergent series.

FEJÉR's idea of investigating the nature of the divergence of a development of an arbitrary function in terms of normal functions and then applying general theorems about convergence factors to determine the behavior of the series when such factors are introduced, can be applied to many other developments that occur in Mathematical Physics. The present paper is devoted to a study, from this point of view, of the developments in terms of Bessel functions.

Although the formal work of obtaining the development of an arbitrary function in terms of Bessel functions goes back to FOURIER, the rigorous discussion of the conditions under which the development is convergent has not made equal progress with that discussion in the case of the ordinary Fourier's series. This is doubtless due to the less elementary character of the functions involved

* Presented to the Society, September 11, 1908.

† Cf. Mathematische Annalen, vol. 58 (1903-04), p. 51.

‡ The word summable will be used in this article in a somewhat special sense. Cf. the definition in § 2.

and to the fact that the formulæ in the case of the Bessel functions are considerably more cumbersome. As far as I am able to learn the first and almost the only satisfactory investigation of the convergence of the development to the value of the function developed is due to DINI.* KNESER has discussed,† from the standpoint of integral equations, the convergence of the development in terms of J_0, and has stated that similar methods will apply to the developments in terms of J_ν where $\nu > 0$. A number of other discussions of the convergence of the development have been published, but I do not know of any treatment, besides those mentioned, in which the reasoning is complete.

The discussion of the summability of the development, given in this paper, is analogous in its general outlines to the discussion of the convergence given by HARNACK.‡ HARNACK's treatment is not adequate but his method is suggestive.

§ 1. *Distribution of the roots of an equation involving Bessel functions.*

We wish to discuss in this section the distribution of the positive roots of the equation

$$l\lambda J'_\nu(\lambda) + hJ_\nu(\lambda) = 0, \tag{1}$$

where $J_\nu(\lambda)$ represents a Bessel function of the νth order, ν is a constant positive or zero, and h and l are any constants not both zero.

We find it convenient to consider first the following trigonometric equation

$$\cos(\lambda - a) + \frac{k}{\lambda}\sin(\lambda - a) = 0 \qquad (k, a \text{ constants}) \tag{2}$$

whose roots will be found to approximate to the roots of equation (1) for large values of λ.

LEMMA 1. *If we represent by* $\lambda'_1, \lambda'_2, \lambda'_3, \cdots$ *the successive positive roots of equation* (2), *we have*

$$|\lambda'_{m+1} - \lambda'_m - \pi| < \frac{K_1}{\lambda'^2_m} \qquad (m = 1, 2, 3, \cdots) \tag{3}$$

where K_1 *is a positive constant.*

This lemma is obviously true if $k = 0$. Let us then first consider the case in which $k > 0$, and let λ_0 be any positive value of λ such that

$$\cos(\lambda_0 - a) = 0, \qquad \sin(\lambda_0 - a) = -1, \tag{4}$$

$$\lambda_0 > k. \tag{5}$$

* *Serie di Fourier* (Pisa, 1880), pp. 246–269.

† Mathematische Annalen, vol. 63 (1906–07), p. 505.

‡ Mathematische Annalen, vol. 35 (1890), p. 41.

Since, in view of (4), $\cos(\lambda - a)$ steadily increases from 0 to $\frac{1}{2}\sqrt{3}$ and $\sin(\lambda - a)$ steadily increases from -1 to $-\frac{1}{2}$ while λ increases from λ_0 to $\lambda_0 + \frac{1}{3}\pi$, it follows that the expression on the left hand side of equation (2), which is negative for $\lambda = \lambda_0$ and, in view of (5), is positive for $\lambda = \lambda_0 + \frac{1}{3}\pi$, must vanish at one and only one point in the interval $\lambda_0 \leqq \lambda \leqq \lambda_0 + \frac{1}{3}\pi$. Moreover, this expression is always positive in the interval $\lambda_0 + \frac{1}{3}\pi \leqq \lambda \leqq \lambda_0 + \pi$, and hence equation (2) has one and only one root in the interval $\lambda_0 \leqq \lambda \leqq \lambda_0 + \pi$.

If we denote this root by λ'_m, we have

$$\begin{aligned}\cos(\lambda'_m + \pi - a) + \frac{k}{\lambda'_m + \pi}\sin(\lambda'_m + \pi - a) \\ = \frac{k}{\lambda'_m}\sin(\lambda'_m - a) + \frac{k}{\lambda'_m + \pi}\sin(\lambda'_m + \pi - a) \\ = \frac{\pi k \sin(\lambda'_m - a)}{\lambda'_m(\lambda'_m + \pi)} < 0. \end{aligned} \tag{6}$$

Since $\cos(\lambda - a) + (k/\lambda)\sin(\lambda - a)$ is positive for $\lambda = \lambda_0 + \pi$, it follows from (6) that λ'_{m+1} lies in the interval $\lambda_0 + \pi < \lambda < \lambda'_m + \pi$. Moreover we have from (6)

$$\begin{aligned}&\frac{\pi k \sin(\lambda'_m + \pi - a)}{\lambda'_m(\lambda'_m + \pi)} \\ &= \cos(\lambda'_{m+1} - a) + \frac{k}{\lambda'_{m+1}}\sin(\lambda'_{m+1} - a) - \cos(\lambda'_m + \pi - a) - \frac{k}{\lambda'_m + \pi}\sin(\lambda'_m + \pi - a) \\ &> \cos(\lambda'_{m+1} - a) - \cos(\lambda'_m + \pi - a) + \frac{k}{\lambda'_{m+1}}\sin(\lambda'_m + \pi - a) \\ &\qquad - \frac{k}{\lambda'_m + \pi}\sin(\lambda'_m + \pi - a) \\ &= \cos(\lambda'_{m+1} - a) - \cos(\lambda'_m + \pi - a) + k\sin(\lambda'_m + \pi - a)\left[\frac{\lambda'_m + \pi - \lambda'_{m+1}}{(\lambda'_m + \pi)\lambda'_{m+1}}\right].\end{aligned} \tag{7}$$

But since from the Law of the Mean

$$\cos(\lambda'_{m+1} - a) - \cos(\lambda'_m + \pi - a) = -\sin\xi \cdot [\lambda'_{m+1} - \lambda'_m - \pi] \qquad (\lambda'_{m+1} - a < \xi < \lambda'_m + \pi - a)$$

we have from (7)

$$-[\lambda'_{m+1} - \lambda'_m - \pi] < \frac{k\sin(\lambda'_m + \pi - a)}{\sin\xi}\left[\frac{\pi}{(\lambda'_m + \pi)\lambda'_m} - \frac{\lambda'_m + \pi - \lambda'_{m+1}}{(\lambda'_m + \pi)\lambda'_{m+1}}\right],$$

and since, moreover,

$$\sin\xi > \sin(\lambda'_m + \pi - a) = -\sin(\lambda'_m - a) > -\sin\left(\lambda_0 + \frac{\pi}{3} - a\right) = \frac{1}{2}$$

and $\lambda'_{m+1} - \lambda'_m - \pi$ is negative, we have finally

$$|\lambda'_{m+1} - \lambda'_m - \pi| < 2k\left[\frac{\pi}{(\lambda'_m + \pi)\lambda'_m} + \frac{\lambda'_m + \pi - \lambda'_{m+1}}{(\lambda'_m + \pi)\lambda'_{m+1}}\right]$$

(8)

$$< 2k\left[\frac{\pi}{\lambda'^2_m} + \frac{\pi}{\lambda'^2_m}\right] = \frac{4\pi k}{\lambda'^2_m}.$$

The inequality (8) has been derived under the assumption that λ_0, the root of $\cos(\lambda - a) = 0$ that immediately precedes λ'_m, is such that $\sin(\lambda_0 - a) = -1$ and the inequality (5) is satisfied. The first restriction was made for definiteness and the proof of formula (8) can be carried through in a similar manner if (4) is replaced by

(4′) $$\cos(\lambda_0 - a) = 0, \qquad \sin(\lambda_0 - a) = 1.$$

Moreover, since there are only a finite number of positive roots of equation (2) lying in the interval $0 < \lambda < \bar{\lambda}_0$ where $\bar{\lambda}_0$ is the least value of λ_0 satisfying (4) and (5), or (4′) and (5), we may choose $K_1 > 4\pi k$ and such that the inequality (3) is satisfied for all positive roots of (2) that are less than $\bar{\lambda}_0$. Hence, in view of (8), we have

$$|\lambda'_{m+1} - \lambda'_m - \pi| < \frac{K_1}{\lambda'^2_m} \qquad (m = 1, 2, 3, \cdots)$$

and the lemma is proved for the case where $k > 0$.

The proof for the case where $k < 0$ is quite analogous, the principal difference being that the successive roots of (2) come before instead of after the corresponding roots of $\cos(\lambda - a) = 0$ to which they approximate.

We will consider next the equation

(9) $$\cos(\lambda - a) + \frac{k}{\lambda}\sin(\lambda - a) + \epsilon(\lambda) = 0$$

where k and a are constants and $\epsilon(\lambda)$ is a continuous function of λ satisfying the inequality

(10) $$|\epsilon(\lambda)| < \frac{M}{\lambda^2} \qquad (M \text{ a positive constant}).$$

LEMMA 2. *Corresponding to each root λ'_m of equation (2) there is a root λ_n of equation (9) such that*

(11) $$|\lambda_n - \lambda'_m| < \frac{K_2}{\lambda'^2_m}$$

where K_2 is a positive constant.

Let us first consider the case where $k > 0$; and let λ_0 be any positive value of λ greater than $\frac{3}{4}\pi$ and such that

(12) $$\cos(\lambda_0 - a) = 0, \qquad \sin(\lambda_0 - a) = -1;$$

(13) $$\frac{k}{\lambda_0 - \frac{1}{6}\pi} < \frac{1}{8}, \qquad \frac{M}{(\lambda_0 - \frac{1}{6}\pi)^2} < \frac{\pi}{48}.$$

Then equation (2) will have one and only one root in the interval $\lambda_0 < \lambda < \lambda_0 + \frac{1}{12}\pi$. Denote that root by λ'_m. For any positive increment $\Delta\lambda$ we have

$$\cos(\lambda'_m + \Delta\lambda - a) + \frac{k}{\lambda'_m + \Delta\lambda}\sin(\lambda'_m + \Delta\lambda - a)$$

$$= \cos(\lambda'_m + \Delta\lambda - a) + \frac{k}{\lambda'_m + \Delta\lambda}\sin(\lambda'_m + \Delta\lambda - a) - \cos(\lambda'_m - a) - \frac{k}{\lambda'_m}\sin(\lambda'_m - a) \tag{14}$$

$$= \cos(\lambda'_m + \Delta\lambda - a) - \cos(\lambda'_m - a) - \frac{k\Delta\lambda}{\lambda'_m(\lambda'_m + \Delta\lambda)}\sin(\lambda'_m + \Delta\lambda - a)$$

$$+ \frac{k}{\lambda'_m}[\sin(\lambda'_m + \Delta\lambda - a) - \sin(\lambda'_m - a)].$$

But

$$\cos(\lambda'_m + \Delta\lambda - a) - \cos(\lambda'_m - a) = -\Delta\lambda \sin(\lambda'_m + \theta\Delta\lambda - a) \quad (0 < \theta < 1),$$

or if we take $\Delta\lambda < \frac{1}{4}\pi$ so that

$$-1 < \sin(\lambda'_m + \theta\Delta\lambda - a) < \sin\left(\lambda_0 + \frac{\pi}{3} - a\right) = -\frac{1}{2},$$

$$\cos(\lambda'_m + \Delta\lambda - a) - \cos(\lambda'_m - a) > \frac{\Delta\lambda}{2}. \tag{14a}$$

Moreover, in view of (13),

$$\left|\frac{k\Delta\lambda}{\lambda'_m(\lambda'_m + \Delta\lambda)}\sin(\lambda'_m + \Delta\lambda - a)\right| < \frac{k}{\lambda'_m}\Delta\lambda < \frac{\Delta\lambda}{8}, \tag{14b}$$

and since

$$\sin(\lambda'_m + \Delta\lambda - a) - \sin(\lambda'_m - a) = \Delta\lambda\cos(\lambda'_m + \theta_1\Delta\lambda - a) \quad (0 < \theta_1 < 1),$$

we have from (13)

$$\frac{k}{\lambda'_m}\cdot|\sin(\lambda'_m + \Delta\lambda - a) - \sin(\lambda'_m - a)| < \frac{\Delta\lambda}{8}. \tag{14c}$$

Combining (14) with (14a), (14b), and (14c), we obtain

$$\cos(\lambda'_m + \Delta\lambda - a) + \frac{k}{\lambda'_m + \Delta\lambda}\sin(\lambda'_m + \Delta\lambda - a) > \frac{\Delta\lambda}{4} \quad \left(0 < \Delta\lambda < \frac{\pi}{4}\right) \tag{15}$$

If now we choose $\Delta\lambda$ so that

$$\Delta\lambda = \frac{4M}{(\lambda_0 - \frac{1}{6}\pi)^2} < \frac{\pi}{12}, \tag{16}$$

we have, since $\lambda'_m + \Delta\lambda > \lambda_0 - \frac{1}{6}\pi$,

$$\frac{\Delta\lambda}{4} = \frac{M}{(\lambda_0 - \frac{1}{6}\pi)^2} > \frac{M}{(\lambda_m + \Delta\lambda)^2},$$

and hence, in view of (10) and (15),

$$\cos(\lambda_m' + \Delta\lambda - a) + \frac{k}{\lambda_m' + \Delta\lambda}\sin(\lambda_m' + \Delta\lambda - a) + \epsilon(\lambda_m' + \Delta\lambda) > 0. \tag{17}$$

In a precisely similar manner it may be shown that

$$\cos(\lambda_m' - \Delta\lambda - a) + \frac{k}{\lambda_m' - \Delta\lambda}\sin(\lambda_m' - \Delta\lambda - a) + \epsilon(\lambda_m' - \Delta\lambda) < 0. \tag{18}$$

It follows from (17) and (18) that equation (9) has at least one root in the interval

$$\lambda_m' - \Delta\lambda < \lambda < \lambda_m' + \Delta\lambda.$$

If we represent such a root by λ_n, we have from (16)

$$\begin{aligned} |\lambda_n - \lambda_m'| &< \frac{4M}{(\lambda_0 - \frac{1}{6}\pi)^2} < \frac{4M}{(\lambda_m' - \frac{1}{4}\pi)^2} = \frac{4M}{\lambda_m'^2}\cdot\frac{\lambda_m'^2}{(\lambda_m' - \frac{1}{4}\pi)^2} \\ &= \frac{4M}{\lambda_m'^2}\cdot\frac{1}{\left(1 - \frac{\pi}{4\lambda_m'}\right)^2} < \frac{9M}{\lambda_m'^2}. \end{aligned} \tag{19}$$

The inequality (19) has been derived under the assumption that λ_0, the root of $\cos(\lambda - a) = 0$ that immediately precedes λ_m', is such that $\sin(\lambda_0 - a) = -1$ and the inequalities (13) are satisfied. The first restriction was made for definiteness and the proof can be carried through in a similar manner if (12) is replaced by

$$\cos(\lambda_0 - a) = 0, \qquad \sin(\lambda_0 - a) = 1. \tag{12'}$$

Moreover, since there are only a finite number of roots of equation (2) lying in the interval $0 < \lambda < \bar{\lambda}_0$ where $\bar{\lambda}_0$ is the least value of λ_0 satisfying (12) and (13), or (12') and (13), we may choose K_2 greater than $9M$ and such that the inequality (11) is satisfied for all the positive roots of (2) that are less than $\bar{\lambda}_0$, provided we choose as the λ_n corresponding to each λ_m' that root of (9) which lies nearest to λ_m'. Hence we have, in view of (19),

$$|\lambda_n - \lambda_m'| < \frac{K_2}{\lambda_m'^2} \qquad (m = 1, 2, 3, \cdots),$$

and the lemma is proved for the case of $k > 0$. The proof for the case of $k < 0$ is quite analogous, and for $k = 0$ the proof is simpler.

We will now show the relationship between equation (9) and equation (1) by proving the following lemma:

LEMMA 3. *The positive roots of any equation of the form* (1) *are the same as the positive roots of a corresponding equation of the form* (9).

We obtain easily from the asymptotic expansion of $J_\nu(\lambda)$*

$$(20)\quad J_\nu(\lambda)=\sqrt{\frac{2}{\pi\lambda}}\left[\cos\left(\lambda-\frac{2\nu+1}{4}\pi\right)+\frac{1-4\nu^2}{8\lambda}\sin\left(\lambda-\frac{2\nu+1}{4}\pi\right)+\epsilon_\nu(\lambda)\right]$$

where

$$|\epsilon_\nu(\lambda)|<\frac{M_1}{\lambda^2}\qquad(\lambda>0),$$

M_1 being a positive constant.

We have for the derivative of $J_\nu(\lambda)$†

$$J_\nu'(\lambda)=\frac{\nu}{\lambda}J_\nu(\lambda)-J_{\nu+1}(\lambda)$$

and on substituting this value in equation (1) we get

$$(21)\qquad (\nu l+h)J_\nu(\lambda)-l\lambda J_{\nu+1}(\lambda)=0.$$

This equation may be thrown into another form by replacing $J_{\nu+1}(\lambda)$ by the value obtained by changing ν into $\nu+1$ in (20) and also replacing $J_\nu(\lambda)$ by

$$-\sqrt{\frac{2}{\pi\lambda}}\left[\sin\left(\lambda-\frac{2\nu+3}{4}\pi\right)+\bar{\epsilon}_\nu(\lambda)\right]\qquad\left(|\bar{\epsilon}_\nu(\lambda)|<\frac{M_2}{\lambda}\right),$$

a value readily obtained from (20). We thus reduce (21) to the form

$$(22)\quad -l\sqrt{\frac{2\lambda}{\pi}}\left[\cos\left(\lambda-\frac{2\nu+3}{4}\pi\right)+\frac{8(\nu l+h)-l(4\nu^2+8\nu+3)}{8l\lambda}\sin\left(\lambda-\frac{2\nu+3}{4}\pi\right)+\epsilon(\lambda)\right]=0$$

where

$$|\epsilon(\lambda)|<\frac{M}{\lambda^2},$$

M being a positive constant.

Since the asymptotic expansions hold for all positive values of the argument, it follows that equation (22) is equivalent to equation (21) for all positive values of λ, and hence the positive roots of any equation of the form (1) are the same as the positive roots of an equation of the form (9).‡

We are now ready to prove the following theorem:

THEOREM I. *If we represent by* $\lambda_1, \lambda_2, \lambda_3, \cdots$ *the successive positive roots of equation* (1), *we have*

$$(23)\qquad \lambda_n>Cn,\qquad |\lambda_{n+1}-\lambda_n-\pi|<\frac{K}{n^2}$$

where C *and* K *are positive constants.*

*Cf. LIPSCHITZ, Crelle's Journal, vol. 56 (1859), pp. 193–196.

†Cf. GRAY and MATHEWS, *Treatise on Bessel Functions*, p. 13.

‡There would seem to be an exception for the case $l=0$. However, in that case (21) reduces by the substitution of (20) to equation (26) given below which is of the form of equation (9).

We prove this theorem first in the special case $l = 0$, in which case (1) reduces to

$$J_\nu(\lambda) = 0. \tag{24}$$

STURM has shown * that for this case

$$\lim_{n=\infty} (\lambda_{n+1} - \lambda_n) = \pi. \tag{25}$$

From this fact the first inequality (23) follows at once.

In view of (20), equation (24) may be written for positive values of λ

$$\cos\left(\lambda - \frac{2\nu+1}{4}\pi\right) + \frac{1-4\nu^2}{8\lambda}\sin\left(\lambda - \frac{2\nu+1}{4}\pi\right) + \epsilon_\nu(\lambda) = 0 \quad \left(|\epsilon_\nu(\lambda)| < \frac{M_1}{\lambda^2}\right). \tag{26}$$

Hence, however small the positive quantity δ, all the large roots of (24) lie in the intervals

$$\left|\frac{2\nu+1}{4}\pi + \frac{\pi}{2} + k\pi - \lambda\right| < \delta \qquad (k = 1, 2, \cdots). \tag{27}$$

Since the distance between two successive roots of (24) approaches π as its limit, it is clear that *one and only one* of these roots lies in each of the more distant of the intervals (27).

Precisely the same reasoning applies to the roots of the equation

$$\cos\left(\lambda - \frac{2\nu+1}{4}\right)\pi + \frac{1-4\nu^2}{8\lambda}\sin\left(\lambda - \frac{2\nu+1}{4}\pi\right) = 0, \tag{28}$$

since the distance between two successive roots here also approaches π as its limit, as we see from Lemma 1.

Hence all the large roots of (26) coincide very nearly with the large roots of (28) and vice versa.

Let us now denote by λ_n' that positive root of (28) which nearly coincides with the large positive root λ_n of (26).† Consequently $\lambda_n' > C'n$. By this inequality and Lemmas 1 and 2 we have for all large values of n

$$|\lambda_n - \lambda_n'| < \frac{K_2}{\lambda_n'^2} < \frac{K_2'}{n^2}, \qquad |\lambda_{n+1}' - \lambda_n' - \pi| < \frac{K_1}{\lambda_n'^2} < \frac{K_1'}{n^2}.$$

For these large values of n we therefore have

$$\begin{aligned} |\lambda_{n+1} - \lambda_n - \pi| &= |(\lambda_{n+1} - \lambda_{n+1}') - (\lambda_n - \lambda_n') + (\lambda_{n+1}' - \lambda_n' - \pi)| \\ &\leqq |\lambda_{n+1} - \lambda_{n+1}'| + |\lambda_n - \lambda_n'| + |\lambda_{n+1}' - \lambda_n' - \pi| < \frac{2K_2' + K_1'}{n^2}. \end{aligned}$$

* Liouville's Journal, vol. 1 (1836), pp. 174–175.

† Notice that λ_n' is not necessarily the nth positive root of (28). However, λ_n' and λ_{n+1}' are *successive* positive roots of (28), and this is all that is essential.

If then we take $K > 2K_2' + K_1'$, we see that the second inequality (23) is established for all large values of n. It is clear that K may be increased so that it will also hold for the smaller positive values of n. Thus our theorem is proved when $l = 0$.

Passing now to the case $l \neq 0$, we note first that by a theorem of STURM* between two successive roots of (24) there lies one and only one root of (1). From this, together with the fact that the first inequality (23) holds when $l = 0$, we infer at once the truth of this formula when $l \neq 0$.

Reasoning now on equation (22) precisely as we reasoned above on (26), we see that however small δ may be chosen, all the large roots of (1) lie in the intervals

$$\left| \frac{2\nu + 3}{4}\pi + \frac{\pi}{2} + k\pi - \lambda \right| < \delta \qquad (k = 1, 2, \cdots), \tag{29}$$

and since these intervals for small values of δ alternate with the intervals (27) we see from the theorem of STURM last referred to that for large values of k *one and only one* root of (1) lies in each interval (29).

That furthermore the large roots of

$$\cos\left(\lambda - \frac{2\nu + 3}{4}\pi\right) + \frac{8(\nu l + h) - l(4\nu^2 + 8\nu + 3)}{8l\lambda} \sin\left(\lambda - \frac{2\nu + 3}{4}\pi\right) = 0 \tag{30}$$

also all lie in the intervals (29) is obvious; that one and only one of them lies in each of these intervals follows from the fact that the distance between two successive roots approaches π as its limit, as we see from Lemma 1.

Hence all the large roots of (30) coincide very nearly with the large roots of (22).

From here on the reasoning is identical with the reasoning in the case $l = 0$; and our theorem is established.

COROLLARY. *If we represent by* $\lambda_1, \lambda_2, \lambda_3, \cdots$ *the successive positive roots of equation* (1), *then there exists a constant* q *and a positive constant* L *such that*

$$|\lambda_n - n\pi - q| < \frac{L}{n} \qquad (n = 1, 2, 3, \cdots). \tag{31}$$

Let us consider the series

$$\sum_{n=1}^{\infty} [\lambda_{n+1} - \lambda_n - \pi], \tag{32}$$

which by Theorem I is absolutely convergent.

If we represent the value to which it converges by q_1 and its remainder after $(n-1)$ terms by R_n, we have

$$\lambda_n - \lambda_1 - (n-1)\pi + R_n = q_1,$$

* Loc. cit., § XIX.

and consequently

$$|\lambda_n - n\pi + \pi - \lambda_1 - q_1| = |R_n| < \sum_{n=n}^{\infty} \frac{K}{n^2}$$

$$< \int_{n-1}^{\infty} \frac{K}{x^2}\, dx = \frac{K}{n} \cdot \frac{n}{n-1} \leqq \frac{2K}{n} \qquad (n \geqq 2).$$

Hence if we take $q = q_1 + \lambda_1 - \pi$, and $L \geqq 2K$ sufficiently large to satisfy the inequality (31) for $n = 1$, the truth of our corollary follows.

§ 2. *Summability and uniform summability of the development of an arbitrary function in terms of Bessel functions.*

We begin this section with a definition of summability and uniform summability and the statement of a few general facts with regard to summable and uniformly summable series.

DEFINITION. *If $u_1(x), u_2(x), \cdots$ are functions of x, and we let*

$$s_n(x) = u_1(x) + u_2(x) + \cdots + u_n(x), \tag{33}$$

$$S_n(x) = \frac{s_1(x) + s_2(x) + \cdots + s_n(x)}{n}, \tag{34}$$

then if for a particular value of x, $S_n(x)$ approaches a limit $U(x)$ as n becomes infinite, the series

$$u_1(x) + u_2(x) + \cdots$$

*is said to be summable to the value $U(x)$; if $S_n(x)$ approaches $U(x)$ uniformly throughout a certain interval, the series is said to be uniformly summable.**

It is a well known fact that every convergent series is summable, and every uniformly convergent series is uniformly summable, to the values to which they are convergent or uniformly convergent, respectively, but that the converse is not true. Moreover, the sum of any two summable or uniformly summable series is summable or uniformly summable, respectively, and its value is equal to the sum of the values of the original series.

We will now proceed to establish some sufficient conditions that the series

$$\sum_{n=1}^{\infty} J_\nu(\lambda_n x) \frac{\int_0^1 x f(x) J_\nu(\lambda_n x)\, dx}{\int_0^1 x [J_\nu(\lambda_n x)]^2\, dx}, \tag{35}$$

where $\lambda_1, \lambda_2, \cdots$ are the successive positive roots of equation (1), should be

* The term *summable* is often used in other more general senses. With these we shall not be concerned in this paper.

summable at every point of the interval $0 < x < 1$ at which the function $f(x)$, which appears in the formulæ for the coefficients, is continuous or has a finite jump, and should be uniformly summable throughout every closed sub-interval lying within an interval of continuity of $f(x)$. The series (35) is the ordinary formal development of an arbitrary function $f(x)$ in terms of Bessel functions.

We begin by proving two lemmas.

LEMMA 4. *If we represent by* $\lambda_1, \lambda_2, \cdots$ *the successive positive roots of equation* (1), *then the series*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\rho}} \int_0^1 \phi(x) \sin(\lambda_n x - \alpha)\, dx^* \qquad (\rho > \tfrac{1}{2}), \tag{36}$$

where α *is any constant, will be absolutely convergent, provided* $\phi(x)$ *is finite and integrable.*

Using the Law of the Mean and the corollary of Theorem I we obtain the formula

$$\left| \int_0^1 \phi(x) \sin(\lambda_n x - \alpha)\, dx - \int_0^1 \phi(x) \sin\{(n\pi + q)x - \alpha\}\, dx \right|$$
$$= \left| \int_0^1 \phi(x)(\cos \xi)(\lambda_n - n\pi - q)x\, dx \right| < \frac{L}{n} \int_0^1 |\phi(x)|\, dx = \frac{L_1}{n},$$

where L_1 is a positive constant. We may therefore write

$$\begin{aligned} &\left| \frac{1}{\lambda_n^{\rho}} \int_0^1 \phi(x) \sin(\lambda_n x - \alpha)\, dx \right| \\ &\quad \leqq \frac{1}{\lambda_n^{\rho}} \left\{ \left| \int_0^1 \phi(x) [\sin(\lambda_n x - \alpha) - \sin\{(n\pi + q)x - \alpha\}]\, dx \right| \right. \\ &\qquad \left. + \left| \int_0^1 \phi(x) \sin\{(n\pi + q)x - \alpha\}\, dx \right| \right\} \\ &\quad < \frac{L_1}{n\lambda_n^{\rho}} + \frac{1}{\lambda_n^{\rho}} \left| \int_0^1 \phi(x) \sin\{(n\pi + q)x - \alpha\}\, dx \right|. \end{aligned} \tag{37}$$

But by Theorem I

$$\frac{L_1}{n\lambda_n^{\rho}} < \frac{L_1}{C^{\rho} n^{1+\rho}},$$

and hence the series whose general term is the first term on the right hand side of (37) is convergent. If now we can show that the series which has for its general term the second term on the right hand side of (37) is convergent, it will follow that the series (36) is absolutely convergent.

* We have taken the sine as the trigonometric function appearing in the integrand. That the lemma holds equally well for the cosine is seen by making the change of variable

$$\alpha = \alpha' - \frac{\pi}{2}.$$

We have

$$\int_0^1 \phi(x) \sin\{(n\pi + q)x - \alpha\}\, dx = u_n + v_n \tag{38}$$

where

$$u_n = \int_0^1 \phi(x) \cos(qx - \alpha) \sin n\pi x\, dx, \qquad v_n = \int_0^1 \phi(x) \sin(qx - \alpha) \cos n\pi x\, dx.$$

These quantities u_n and v_n are constant multiples of Fourier's constants belonging to the functions $\phi(x)\cos(qx-\alpha)$ and $\phi(x)\sin(qx-\alpha)$, respectively. Therefore, since these functions are finite and integrable, it follows from a theorem due to DE LA VALLÉE POUSSIN that

$$\sum_{n=1}^{\infty} u_n^2, \qquad \sum_{n=1}^{\infty} v_n^2$$

are convergent.* Hence the series

$$\sum_{n=1}^{\infty} \frac{|u_n|}{n^\rho}, \qquad \sum_{n=1}^{\infty} \frac{|v_n|}{n^\rho} \qquad (\rho > \tfrac{1}{2}), \tag{39}$$

are convergent.†

But we have from (38) and Theorem I

$$\frac{1}{\lambda_n^\rho}\left|\int_0^1 \phi(x) \sin\{(n\pi + q)x - \alpha\}\, dx\right| \leqq \frac{|u_n|}{\lambda_n^\rho} + \frac{|v_n|}{\lambda_n^\rho} \leqq \frac{|u_n|}{C^\rho n^\rho} + \frac{|v_n|}{C^\rho n^\rho}. \tag{40}$$

Since the two series in (39) are convergent, it follows from (40) that the series whose general term is the left hand side of (40) is convergent, and our lemma is proved.

LEMMA 5. *If the general term of the series* (35) *is written in the form*

$$\begin{aligned} &\frac{2}{\sqrt{x}} \cos(qx - \alpha) \cos n\pi x \int_0^1 \sqrt{x} f(x) \cos(qx - \alpha) \cos n\pi x\, dx \\ &+ \frac{2}{\sqrt{x}} \sin(qx - \alpha) \sin n\pi x \int_0^1 \sqrt{x} f(x) \sin(qx - \alpha) \sin n\pi x\, dx + v_n(x), \end{aligned} \tag{41}$$

then, provided the function $f(x)$ is finite and integrable, the series

$$\sum_{n=1}^{\infty} v_n(x)$$

will be uniformly summable in every interval

$$c \leqq x \leqq 1 \tag{42}$$

*Annales de la Société Scientifique de Bruxelles, vol. 17 (1892-3); also BÔCHER, Annals of Mathematics, ser. 2, vol. 7 (1906), p. 107.

† BÔCHER, loc. cit., p. 108, lemma. This lemma is there proved only for the case $\rho = 1$ but the proof applies equally well to the general case $\rho > \frac{1}{2}$.

for which

$$0 < c < 1,$$

and if we write

$$\bar{v}_n(x) = x^{\frac{1}{2}} v_n(x) \qquad (0 < x \leqq 1),$$

$$\bar{v}_n(0) = \lim_{x=+0} \bar{v}_n(x)$$

the series

$$\sum_{n=1}^{\infty} \bar{v}_n(x)$$

will be uniformly summable and its terms will be continuous in the interval

$$0 \leqq x \leqq 1. \tag{43}$$

It is obvious that it will be sufficient to prove only the second part of the above lemma, since the first part is an immediate consequence of the second. We will show then that the facts stated hold with regard to the series whose general term is $\bar{v}_n(x)$, and it will follow at once that the facts stated hold also with regard to the series whose general term is $v_n(x)$.

In view of the asymptotic expansion of $J_\nu(\lambda)$ [see equation (20)] we may write

$$J_\nu(\lambda) = \frac{A}{\sqrt{\lambda}} \cos(\lambda - \alpha) + \frac{\psi(\lambda)}{\lambda^{\frac{3}{2}}}, \qquad J_{\nu+1}(\lambda) = \frac{A}{\sqrt{\lambda}} \sin(\lambda - \alpha) + \frac{\psi(\lambda)}{\lambda^{\frac{3}{2}}}, \tag{44}$$

where A and α are constants and *we use* $\psi(\lambda)$ *to represent any function of* λ *that is continuous for all positive values of* λ, *remains finite for all such values of* λ, *and approaches a finite limit when* λ *approaches zero.* It is obvious from this definition that in different formulas, and even in different parts of the same formula, $\psi(\lambda)$ may have a different meaning.

If we set $\lambda x = y$ we obtain

$$\int_0^1 x\,[J_\nu(\lambda x)]^2\,dx = \frac{1}{\lambda^2} \int_0^\lambda y\,[J_\nu(y)]^2\,dy. \tag{45}$$

But we have*

$$\int_0^\lambda y\,[J_\nu(y)]^2\,dy = \frac{\lambda^2}{2}\{[J_\nu(\lambda)]^2 + [J_{\nu+1}(\lambda)]^2\} - \nu\lambda J_\nu(\lambda) J_{\nu+1}(\lambda),$$

and hence we obtain from (44) and (45) as a value for the denominator of the fraction in (35)†

$$\int_0^1 x\,[J_\nu(\lambda x)]^2\,dx = \frac{A^2}{2\lambda} + \frac{\psi_1(\lambda)}{\lambda^2}, \tag{46}$$

where $\psi_1(\lambda)$ *is used to represent any function of* λ *that remains finite when*

* Cf. BYERLY, *An Elementary Treatise on Fourier's Series, etc.*, p. 224.

† For the sake of simplicity we use λ instead of λ_n.

λ *increases indefinitely.* It is apparent from this definition that a $\psi(\lambda)$ is always a $\psi_1(\lambda)$, but that a $\psi_1(\lambda)$ may not be a $\psi(\lambda)$.

Turn now to the numerator of the fraction in (35). If we use c to denote any constant lying in the interval $0 < x < 1$, and set $\lambda x = y$, we obtain for $\lambda > c$

$$\int_0^{\frac{c}{\lambda}} xf(x)J_\nu(\lambda x)\,dx = \frac{1}{\lambda^2}\int_0^c y f\left(\frac{y}{\lambda}\right)J_\nu(y)\,dy = \frac{\psi_1(\lambda)}{\lambda^2}. \tag{47}$$

Moreover, we have from the asymptotic expansion [see equation (20)]

$$J_\nu(\lambda x) = \frac{A}{\sqrt{\lambda x}}\left[\cos(\lambda x - \alpha) + \frac{1-4\nu^2}{8\lambda x}\sin(\lambda x - \alpha) + \frac{\psi(\lambda x)}{\lambda^2 x^2}\right]. \tag{48}$$

From (48) we obtain

$$\begin{aligned}\int_{\frac{c}{\lambda}}^1 xf(x)J_\nu(\lambda x)\,dx &= \frac{A}{\sqrt{\lambda}}\int_{\frac{c}{\lambda}}^1 \sqrt{x}\,f(x)\cos(\lambda x-\alpha)\,dx \\ &\quad + \frac{A'}{\lambda^{\frac{3}{2}}}\int_{\frac{c}{\lambda}}^1 \frac{f(x)}{\sqrt{x}}\sin(\lambda x-\alpha)\,dx + \frac{A}{\lambda^{\frac{1}{2}}}\int_{\frac{c}{\lambda}}^1 \frac{\psi(\lambda x)f(x)}{x^{\frac{3}{2}}(\lambda x)^{\frac{1}{2}}}\,dx \\ &= \frac{A}{\sqrt{\lambda}}\int_0^1 \sqrt{x}\,f(x)\cos(\lambda x-\alpha)\,dx + \frac{A'}{\lambda^{\frac{3}{2}}}\int_0^1 \frac{f(x)}{\sqrt{x}}\sin(\lambda x-\alpha)\,dx \\ &\quad - \frac{A}{\sqrt{\lambda}}\int_0^{\frac{c}{\lambda}} \sqrt{x}\,f(x)\cos(\lambda x-\alpha)\,dx - \frac{A'}{\lambda^{\frac{3}{2}}}\int_0^{\frac{c}{\lambda}} \frac{f(x)}{\sqrt{x}}\sin(\lambda x-\alpha)\,dx \\ &\quad + \frac{A}{\lambda^{\frac{1}{2}}}\int_{\frac{c}{\lambda}}^1 \frac{\psi(\lambda x)f(x)}{x^{\frac{3}{2}}(\lambda x)^{\frac{1}{2}}}\,dx \\ &= \frac{A}{\sqrt{\lambda}}\int_0^1 \sqrt{x}\,f(x)\cos(\lambda x-\alpha)\,dx + \frac{A'}{\lambda^{\frac{3}{2}}}\int_0^1 \frac{f(x)}{\sqrt{x}}\sin(\lambda x-\alpha)\,dx + \frac{\psi_1(\lambda)}{\lambda^{\frac{1}{2}}}.\end{aligned} \tag{49}$$

Combining (47) and (49) we get

$$\begin{aligned}\int_0^1 xf(x)J_\nu(\lambda x)\,dx &= \frac{A}{\sqrt{\lambda}}\int_0^1 \sqrt{x}\,f(x)\cos(\lambda x-\alpha)\,dx \\ &\quad + \frac{A'}{\lambda^{\frac{3}{2}}}\int_0^1 \frac{f(x)}{\sqrt{x}}\sin(\lambda x-\alpha)\,dx + \frac{\psi_1(\lambda)}{\lambda^{\frac{1}{2}}}.\end{aligned} \tag{50}$$

Hence, for the general term of (35) we obtain by combining (44), (46), and (50)

$$\frac{J_\nu(\lambda x)\int_0^1 xf(x)J_\nu(\lambda x)\,dx}{\int_0^1 x[J_\nu(\lambda x)]^2\,dx} = \frac{P}{\frac{A^2}{2\lambda} + \frac{\psi_1(\lambda)}{\lambda^2}} \tag{51}$$

where

$$P=\left[\frac{A}{\sqrt{\lambda x}}\cos(\lambda x-\alpha)+\frac{\psi(\lambda x)}{(\lambda x)^{\frac{3}{2}}}\right]\left[\frac{A}{\sqrt{\lambda}}\int_0^1\sqrt{x}f(x)\cos(\lambda x-\alpha)dx\right.$$
$$\left.+\frac{A'}{\lambda^{\frac{3}{2}}}\int_0^1\frac{f(x)}{\sqrt{x}}\sin(\lambda x-\alpha)dx+\frac{\psi_1(\lambda)}{\lambda^{\frac{1}{2}}}\right].$$

If we subtract from the right hand side of (51) the two terms

$$\frac{2}{\sqrt{x}}\cos(\lambda x-\alpha)\int_0^1\sqrt{x}f(x)\cos(\lambda x-\alpha)dx, \tag{52}$$

$$\frac{2A'}{A\lambda\sqrt{x}}\cos(\lambda x-\alpha)\int_0^1\frac{f(x)}{\sqrt{x}}\sin(\lambda x-\alpha)dx, \tag{53}$$

the quantity we have left may be written in the form

$$\frac{\left[\frac{A\psi(\lambda x)}{\lambda^2x^{\frac{3}{2}}}+\frac{\psi(x,\lambda)}{\lambda^2\sqrt{x}}\right]\int_0^1\sqrt{x}f(x)\cos(\lambda x-\alpha)dx+\frac{\psi(\lambda x)\psi_1(\lambda)}{\lambda^3x^{\frac{3}{2}}}+\frac{\psi(x,\lambda)}{\lambda^2\sqrt{x}}}{\frac{A^2}{2\lambda}+\frac{\psi_1(\lambda)}{\lambda^2}} \tag{54}$$

where $\psi(x,\lambda)$ *is used to represent any function of* x *and* λ *which for all fixed values of* λ *is continuous in the interval* (43) *and which remains finite when* λ *increases indefinitely and* x *varies in the interval* (43).

We have now broken up the general term of the series (35) into the sum of the three expressions (52), (53), and (54). It is obvious that (53) and (54), when each is multiplied by $x^{\frac{3}{2}}$ and properly defined* for $x=0$, are continuous in the interval (43); we shall show further that each of these expressions when multiplied by $x^{\frac{3}{2}}$ forms the general term of a series which is uniformly summable in that interval. Then we shall prove that if we subtract from (52) a number of expressions which when multiplied by $x^{\frac{3}{2}}$ and properly defined for $x=0$ are continuous in the interval (43) and form the general terms of series that are uniformly summable in that interval, the expression we have left will be equal to the sum of the first and second terms of (41). Our lemma will thus be established.

We will begin with the proof that (54) when multiplied by $x^{\frac{3}{2}}$ forms the general term of a series that is uniformly summable in the interval (43). For values of λ so great that

$$\left|\frac{\psi_1(\lambda)}{\lambda^2}\right|<\frac{A^2}{4\lambda},$$

where the $\psi_1(\lambda)$ occurring in this inequality is the same as the $\psi_1(\lambda)$ in the denominator of (54), the expression (54) when multiplied by $x^{\frac{3}{2}}$ is less in absolute value than

* I. e., defined to have the value it approaches when x approaches zero.

$$\frac{4\lambda}{A^2}\left[\left|\left(\frac{A\psi(\lambda x)}{\lambda^2}+\frac{x\psi(x,\lambda)}{\lambda^2}\right)\int_0^1\sqrt{x}f(x)\cos(\lambda x-\alpha)\,dx\right|\right. \tag{55}$$
$$\left.+\left|\frac{\psi(\lambda x)\psi_1(\lambda)}{\lambda^3}\right|+\left|\frac{x\psi(x,\lambda)}{\lambda^2}\right|\right].$$

Since $\sqrt{x}f(x)$ is finite and integrable in the interval $0 \leqq x \leqq 1$ it follows from Lemma 4 and the first inequality in (23) that the series whose general term is (55) is absolutely and uniformly convergent and therefore uniformly summable in the interval (43).

We will now show that (53) when multiplied by $x^{\frac{1}{2}}$ is the general term of a series that is uniformly summable in the interval (43). If we set $M = 2A'/A$, (53) when multiplied by $x^{\frac{1}{2}}$ becomes

$$\frac{Mx\cos(\lambda x-\alpha)}{\lambda}\int_0^c\frac{f(x)}{\sqrt{x}}\sin(\lambda x-\alpha)\,dx \tag{56}$$
$$+\frac{Mx\cos(\lambda x-\alpha)}{\lambda}\int_c^1\frac{f(x)}{\sqrt{x}}\sin(\lambda x-\alpha)\,dx.$$

The second term in (56) is equal to

$$\frac{Mx\cos(\lambda x-\alpha)}{\lambda}\int_0^1 F(x)\sin(\lambda x-\alpha)\,dx,$$

where we define

$$F(x)=\frac{f(x)}{\sqrt{x}}\quad(c\leqq x\leqq 1),\qquad F(x)=0\quad(0\leqq x<c),$$

and hence it follows from Lemma 4 that it is the general term of a series that is absolutely and uniformly convergent and therefore uniformly summable in the interval (43).

We have next to examine the series whose general term is the first term in (56). It follows from the corollary of Theorem I that

$$\lambda_n = n\pi + q + \frac{\psi_1(\lambda_n)}{n} \tag{57}$$

and consequently

$$\cos(\lambda_n x-\alpha)=\cos\left[(n\pi+q)x-\alpha\right]\cos\frac{x\psi_1(\lambda_n)}{n} \tag{58}$$
$$-\sin\left[(n\pi+q)x-\alpha\right]\sin\frac{x\psi_1(\lambda_n)}{n}$$
$$=\cos\left[(n\pi+q)x-\alpha\right]+\frac{\psi(x,\lambda_n)}{n}.$$

In a similar manner we may also obtain from (57)

$$\sin(\lambda_n x - \alpha) = \sin[(n\pi + q)x - \alpha] + \frac{\psi(x, \lambda_n)}{n}. \tag{59}$$

By the use of (58) and (59) and the first inequality in (23), the first term in (56) reduces to

$$\frac{Mx \cos[(n\pi + q)x - \alpha]}{\lambda_n} \int_0^c \frac{f(x)}{\sqrt{x}} \sin[(n\pi + q)x - \alpha]\, dx + \frac{\psi(x, \lambda_n)}{n^2} \tag{60}$$

and since the second term in (60) is the general term of a series that is absolutely and uniformly convergent and therefore uniformly summable in the interval (43), we have only to examine the series whose general term is the first term in (60).

Using the value of λ_n given by (57), we see that if we add to the first term in (60) the term

$$\frac{\left(q + \frac{\psi_1(\lambda_n)}{n}\right) Mx \cos[(n\pi + q)x - \alpha]}{n\pi\left(n\pi + q + \frac{\psi_1(\lambda_n)}{n}\right)} \int_0^c \frac{f(x)}{\sqrt{x}} \sin[(n\pi + q)x - \alpha]\, dx,$$

which is easily seen to be the general term of a series that is absolutely and uniformly convergent, and therefore uniformly summable in the interval (43), we obtain the expression

$$\frac{Mx \cos[(n\pi + q)x - \alpha]}{n\pi} \int_0^c \frac{f(x)}{\sqrt{x}} \sin[(n\pi + q)x - \alpha]\, dx. \tag{61}$$

If now we can show that the series which has (61) as its general term is uniformly summable in the interval (43), we shall have proved that the series which has (53) multiplied by $x^{\frac{1}{2}}$ as its general term is uniformly summable in that same interval.

By a familiar trigonometric formula, (61) reduces to the form

$$\frac{Mx}{2n\pi} \int_0^c \frac{f(x')}{\sqrt{x'}} \{\sin[(n\pi + q)(x + x') - 2\alpha] - \sin[(n\pi + q)(x - x')]\}\, dx';$$

which, if we set $K = M/2\pi$, may be written in the form

$$\begin{aligned} &Kx \int_0^c \frac{f(x')}{\sqrt{x'}} \cos[q(x + x') - 2\alpha] \frac{\sin n\pi(x + x')}{n}\, dx' \\ &+ Kx \int_0^c \frac{f(x')}{\sqrt{x'}} \sin[q(x + x') - 2\alpha] \frac{\cos n\pi(x + x')}{n}\, dx' \\ &- Kx \int_0^c \frac{f(x')}{\sqrt{x'}} \cos q(x - x') \frac{\sin n\pi(x - x')}{n}\, dx' \\ &- Kx \int_0^c \frac{f(x')}{\sqrt{x'}} \sin q(x - x') \frac{\cos n\pi(x - x')}{n}\, dx'. \end{aligned} \tag{62}$$

We are going to show that each term in (62) is the general term of a series that is uniformly summable in the interval (43).

If we represent by $s_n(z)$ the sum of the first n terms of the series

$$\sum_{n=1}^{\infty} \frac{\sin nz}{n} \tag{63}$$

we have for values of z such that $\sin(z/2) \neq 0$ *

$$s_p(z) - s_n(z) = \frac{\frac{1}{n+1}\cos\frac{(2n+1)z}{2} - \sum_{\nu=n+1}^{\nu=p-1}\left(\frac{1}{\nu} - \frac{1}{\nu+1}\right)\cos\frac{(2\nu+1)z}{2} - \frac{1}{p}\cos\frac{(2p+1)z}{2}}{2\sin\frac{z}{2}} \tag{64}$$

In an analogous manner we obtain for the series

$$\sum_{n=1}^{\infty} \frac{\cos nz}{n} \tag{65}$$

the relation

$$s_p(z) - s_n(z) = \frac{-\frac{1}{n+1}\sin\frac{(2n+1)z}{2} + \sum_{\nu=n+1}^{\nu=p-1}\left(\frac{1}{\nu} - \frac{1}{\nu+1}\right)\sin\frac{(2\nu+1)z}{2} + \frac{1}{p}\sin\frac{(2p+1)z}{2}}{2\sin\frac{z}{2}} \tag{66}$$

In view of (64) we have for the series (63)

$$|s_p(z) - s_n(z)| < \frac{\frac{1}{n+1} + \sum_{\nu=n+1}^{\nu=p-1}\left(\frac{1}{\nu} - \frac{1}{\nu+1}\right) + \frac{1}{p}}{2\left|\sin\frac{z}{2}\right|} = \frac{1}{(n+1)\cdot\left|\sin\frac{z}{2}\right|} \quad (p = n+1, n+2, \cdots). \tag{67}$$

In an analogous manner we obtain for the series (65) by use of the relation (66) the same inequality (67).

If now we represent by $s_n(x)$ the sum of the first n terms of the series whose general term is the first term in (62), we obtain by means of (67), for all values of x in the interval $0 < x \leqq 1$, the inequality

$$|s_p(x) - s_n(x)| \leqq Kx\int_0^c \left|\frac{f(x')}{\sqrt{x'}}\cos[q(x+x') - 2\alpha]\sum_{\nu=n+1}^{\nu=p}\frac{\sin n\pi(x+x')}{n}\right| dx'$$

* Cf. page 110 of BÔCHER's article referred to above.

$$< K\int_0^c \frac{x}{x+x'}\cdot\frac{|f(x')|}{\sqrt{x'}}\cdot\frac{x+x'}{(n+1)\cdot\left|\sin\dfrac{\pi(x+x')}{2}\right|}dx' < \frac{K_1}{n+1}$$

$$(p=n+1,\ n+2,\ \cdots)$$

where K_1 is a positive constant. For $x=0$ the last inequality obviously holds, since each term of the series we are discussing is equal to zero. Hence it follows that the series whose general term is the first term in (62) is uniformly convergent and hence uniformly summable in the interval (43). In a precisely analogous manner we may establish by use of the relation (67) the fact that the series whose general term is the second term in (62) is uniformly summable in the interval (43).

We will consider next the series whose general term is the fourth term in (62). If we represent by $s_n(x)$ the sum of the first n terms of that series we have, in view of (67),

$$|s_p(x)-s_n(x)| \leqq Kx\int_0^c \left|\frac{f(x')}{\sqrt{x'}}\sin q(x-x')\sum_{\nu=n+1}^{\nu=p}\frac{\cos n\pi(x-x')}{n}\right|dx'$$

$$\leqq Kx\int^c \frac{|f(x')|}{\sqrt{x'}}\cdot\frac{|\sin q(x-x')|}{(n+1)\cdot\left|\sin\dfrac{\pi(x-x')}{2}\right|}dx' < \frac{K_2}{n+1}$$

$$(p=n+1,\ n+2,\ \cdots)$$

for all values of x in the interval (43), K_2 being a positive constant. Hence the series whose general term is the fourth term of (62) is uniformly convergent and therefore uniformly summable in the interval (43).

We have finally to consider the series whose general term is the third term in (62). This term may be written in the form

$$(68)\qquad Kx\cos qx\int_c^0\frac{f(x')}{\sqrt{x'}}\cos qx'\frac{\sin n\pi(x-x')}{n}dx' + Kx\sin qx\int_c^0\frac{f(x')}{\sqrt{x'}}\sin qx'\frac{\sin n\pi(x-x')}{n}dx'.$$

By means of an integration by parts we may reduce the first term of (68) to the form

$$(69)\qquad Kx\cos qx\frac{\sin n\pi x}{n}\int_c^0\frac{f(x')}{\sqrt{x'}}\cos qx'\,dx' + \pi Kx\cos qx\int_c^0 F(x')\cos n\pi(x-x')\,dx',$$

where we have set

$$(70)\qquad F(x') = \int_c^{x'}\frac{f(x')}{\sqrt{x'}}\cos qx'dx' \qquad (c\geqq x'\geqq 0).$$

If we represent by $s_n(x)$ the sum of the first n terms of the series whose general term is the first term in (69), we have, by means of (67), for all values of x in the interval $0 < x \leqq 1$,

$$|s_p(x) - s_n(x)| \leqq K \left| \cos qx \int_c^0 \frac{f(x')}{\sqrt{x'}} \cos qx' dx' \right| \frac{x}{(n+1) \cdot \sin \frac{\pi x}{2}} < \frac{K_3}{n+1}$$

$$(p = n+1, n+2, \cdots),$$

where K_3 is a positive constant. For $x = 0$ the last inequality obviously holds, since each term of the series that we are discussing is equal to zero. Hence it follows that the series whose general term is the first term in (69) is uniformly convergent and therefore uniformly summable in the interval (43).

Let us now consider the series whose general term is the second term in (69). If we set

$$u_n(x) = \int_0^2 \chi(x') \cos n\pi (x - x') dx', \tag{71}$$

where

$$\begin{aligned} \chi(x') &= F(x') && (0 \leqq x' \leqq c), \\ \chi(x') &= 0 && (c < x' \leqq 2), \end{aligned} \tag{72}$$

the second term of (69) may be written in the form

$$-\pi K x \cos qx\, u_n(x). \tag{73}$$

However, it follows from (71) that $u_n(x)$ is the general term of the Fourier development corresponding to the function $\chi(x)$, and since we know from (72) and (70) that $\chi(x')$ is finite and integrable in the interval $0 \leqq x \leqq 2$, it follows from FEJÉR's work* that for the series whose general term is $u_n(x)$,

$$|S_n(x)| < N \qquad (n = 1, 2, 3, \cdots; 0 \leqq x \leqq 2), \tag{74}$$

where $S_n(x)$ is defined as in formula (34) and N is a positive constant. If now ϵ is given, positive and arbitrarily small, we have from (74) for the series whose general term is (73) or the second term in (69),

$$|S_p(x) - S_n(x)| \leqq |S_p(x)| + |S_n(x)| < 2\pi KNx |\cos qx| \leqq \epsilon \tag{75}$$

$$\left(n = 1, 2, 3, \cdots; p = n+1, n+2, \cdots; 0 \leqq x \leqq \frac{\epsilon}{B}\right),$$

where we have set $B = 2\pi KN$. Moreover it follows from another theorem of FEJÉR's† that the series whose general term is $u_n(x)$, and hence the series whose general term is (73) or the second term in (69), is uniformly summable in the interval

$$\frac{\epsilon}{B} \leqq x \leqq 1,$$

* Loc. cit., p. 60.

† Loc. cit., p. 60.

since $\chi(x)$ is continuous in an interval including this interval. Hence we may choose m so large that we have for the latter series,

$$|S_p(x) - S_n(x)| < \epsilon \qquad \left(p > n \geqq m\,;\ \frac{\varepsilon}{B} \leqq x \leqq 1\right). \tag{76}$$

Combining (75) and (76), we see that the series whose general term is the second term in (69) is uniformly summable in the interval (43).

Since both the terms in (69) are the general terms of series that are uniformly summable in the interval (43), it follows that the series whose general term is the first term of (68) is uniformly summable in that interval. In an entirely analogous manner it may be shown that the series whose general term is the second term in (68) is uniformly summable in the interval (43), and hence it follows that the series whose general term is (68), or the third term in (62), is uniformly summable in that interval.

We have now shown that each term in (62) is the general term of a series that is uniformly summable in the interval (43). Hence (62), or (61), is the general term of a series that is uniformly summable in that interval and consequently, as we have pointed out before, (53) when multiplied by $x^{\frac{1}{2}}$ is the general term of a uniformly summable series in the same interval.

It remains now to consider the expression (52). It follows from (58) that if we subtract from (52) a term of the form

$$\frac{2\psi(x, \lambda_n)}{n\sqrt{x}} \int_0^1 \sqrt{x} f(x) \cos(\lambda_n x - \alpha)\, dx, \tag{77}$$

which when multiplied by $x^{\frac{1}{2}}$ and properly defined for $x = 0$ is obviously continuous in the interval (43), and by Lemma 4 is the general term of an absolutely and uniformly convergent and therefore uniformly summable series in that same interval, we have left the expression

$$\frac{2}{\sqrt{x}} \cos[(n\pi + q)x - \alpha] \int_0^1 \sqrt{x} f(x) \cos(\lambda_n x - \alpha)\, dx. \tag{78}$$

Moreover we obtain from (58)

$$\cos(\lambda_n x - \alpha) = \cos[(n\pi + q)x - \alpha] - \frac{x\psi_1(\lambda_n)}{n} \sin[(n\pi + q)x - \alpha] + \frac{\psi(x, \lambda_n)}{n^2}$$

and hence (78) reduces to

$$\begin{aligned} &\frac{2}{\sqrt{x}} \cos[(n\pi + q)x - \alpha] \int_0^1 \sqrt{x} f(x) \cos[(n\pi + q)x - \alpha]\, dx \\ &- \frac{2\psi_1(\lambda_n)}{n\sqrt{x}} \cos[(n\pi + q)x - \alpha] \int_0^1 x^{\frac{1}{2}} f(x) \sin[(n\pi + q)x - \alpha]\, dx + \frac{\psi(x, \lambda_n)}{n^2\sqrt{x}}. \end{aligned} \tag{79}$$

The last term in (79) when multiplied by $x^{\frac{1}{2}}$ and properly defined for $x = 0$

is obviously continuous in the interval (43) and is the general term of a series that is uniformly convergent and therefore uniformly summable in that interval. The second term in (79) when multiplied by $x^{\frac{1}{2}}$ and properly defined for $x = 0$ is obviously continuous in the interval (43); that it is also the general term of a uniformly convergent and therefore uniformly summable series in that interval may be seen by referring to the proof of Lemma 4. It is apparent from the latter part of that proof that a series whose general term has the form

$$\frac{1}{n}\int_0^1 \phi(x)\sin\left[(n\pi+q)x-\alpha\right]dx, \tag{80}$$

where $\phi(x)$ is finite and integrable, is absolutely convergent. The second term of (79) when multiplied by $x^{\frac{1}{2}}$ consists of a factor of the form (80) multiplied by a factor that remains finite for all values of λ, and all values of x in the interval (43), and hence this term when multiplied by $x^{\frac{1}{2}}$ is the general term of a series that is absolutely and uniformly convergent and therefore uniformly summable in the interval (43).

It only remains to consider the first term in (79) which may be broken up into the following sum of four terms:

$$\begin{aligned}(81)\quad &\frac{2}{\sqrt{x}}\cos(qx-\alpha)\cos n\pi x\int_0^1\sqrt{x}f(x)\cos(qx-\alpha)\cos n\pi x\,dx\\ &-\frac{2}{\sqrt{x}}\cos(qx-\alpha)\cos n\pi x\int_0^1\sqrt{x}f(x)\sin(qx-\alpha)\sin n\pi x\,dx\\ &-\frac{2}{\sqrt{x}}\sin(qx-\alpha)\sin n\pi x\int_0^1\sqrt{x}f(x)\cos(qx-\alpha)\cos n\pi x\,dx\\ &+\frac{2}{\sqrt{x}}\sin(qx-\alpha)\sin n\pi x\int_0^1\sqrt{x}f(x)\sin(qx-\alpha)\sin n\pi x\,dx.\end{aligned}$$

The first and fourth terms of (81) are identical with the first and second terms of (41). The second and third terms when multiplied by $x^{\frac{1}{2}}$ and properly defined for $x = 0$ are obviously continuous in the interval (43); if we can show further that they form the general term of a series that is uniformly summable in that interval, our lemma will be completely established.

If we use x' as the variable of integration, the second and third terms of (81) when multiplied by $x^{\frac{1}{2}}$ may be written in the form

$$\begin{aligned}&-x\int_0^1\sqrt{x'}f(x')\sin\left[q(x+x')-2\alpha\right]\sin n\pi(x+x')\,dx'\\ &-x\int_0^1\sqrt{x'}f(x')\sin q(x-x')\sin n\pi(x-x')\,dx'.\end{aligned} \tag{82}$$

Since, however,

$$\sum_{n=1}^{n} \sin nz = \frac{\sin z - \sin(n+1)z + \sin nz}{2 - 2\cos z},$$

$$\frac{\sum_{n=1}^{n}\left(\sum_{n=1}^{n} \sin nz\right)}{n} = \frac{(n+1)\sin z - \sin(n+1)z}{2n(1-\cos z)},$$

we obtain for the series whose general term is the expression (82)

$$\begin{aligned} &S_n(x) \\ &= -x\int_0^1 \sqrt{x'}f(x')\sin\left[q(x+x')-2\alpha\right]\frac{(n+1)\sin\pi(x+x')-\sin(n+1)\pi(x+x')}{4n\sin^2\pi\dfrac{x+x'}{2}}\,dx' \\ &\quad -x\int_0^1 \sqrt{x'}f(x')\sin q(x-x')\frac{(n+1)\sin\pi(x-x')-\sin(n+1)\pi(x-x')}{4n\sin^2\pi\dfrac{x-x'}{2}}\,dx', \end{aligned} \tag{83}$$

where $S_n(x)$ is defined as in formula (34).

We are going to show that as n becomes infinite each of the terms on the right hand side of (83) approaches a limit uniformly for all values of x in the interval (43) and consequently that the series whose general term is (82), or the second and third terms in (81) multiplied by $x^{\frac{3}{2}}$, is uniformly summable in that interval.

Consider first the second term on the right hand side of (83). It may be written

$$\begin{aligned} &-\frac{x}{4}\cdot\frac{n+1}{n}\int_0^1 \frac{\sin q(x-x')}{\sin\pi\dfrac{x-x'}{2}}\sqrt{x'}f(x')\frac{\sin\pi(x-x')}{\sin\pi\dfrac{x-x'}{2}}\,dx' \\ &+\frac{x}{4}\cdot\frac{n+1}{n}\int_0^1 \frac{\sin q(x-x')}{\sin\pi\dfrac{x-x'}{2}}\cdot\frac{\sqrt{x'}f(x')(x-x')}{\sin\pi\dfrac{x-x'}{2}}\cdot\frac{\sin(n+1)\pi(x-x')}{(n+1)\sqrt{x-x'}}\cdot\frac{dx}{\sqrt{x-x'}} \end{aligned} \tag{84}$$

The integrand of the integral in the first term of (84) consists of a function of x and x' that is continuous in both variables multiplied into a function of x' that is finite and integrable in the interval of integration. Hence that integral defines a function of x that is continuous and therefore finite in the interval (43), and consequently the first term in (84) approaches a limit uniformly for all values of x in the interval (43) as n becomes infinite.

The factor

$$\frac{\sin(n+1)\pi(x-x')}{(n+1)\sqrt{x-x'}} \tag{85}$$

which occurs in the integrand of the second term in (84) approaches zero uni-

formly for all values of x and x' as n becomes infinite. For, given ϵ positive and arbitrarily small, we have for values of x and x' such that

$$|x-x'|<\frac{\epsilon^2}{\pi^2},$$

$$\left|\frac{\sin(n+1)\pi(x-x')}{(n+1)\sqrt{x-x'}}\right| \leqq \frac{(n+1)\pi|x-x'|}{(n+1)\sqrt{|x-x'|}} = \pi\sqrt{|x-x'|} < \epsilon.$$

Moreover, we can choose m so large that for all values of x and x' such that

$$|x-x'| \geqq \frac{\epsilon^2}{\pi^2},$$

we have

$$\left|\frac{\sin(n+1)\pi(x-x')}{(n+1)\sqrt{x-x'}}\right| < \epsilon \qquad (n \geqq m).$$

We see then that the integral in the second term in (84) is in absolute value not greater than

$$(86)\quad \int_0^1 \left|\frac{\sin q(x-x')}{\sin\pi\dfrac{x-x'}{2}}\cdot\frac{\sqrt{x'}f(x')(x-x')}{\sin\pi\dfrac{x-x'}{2}}\right|\cdot\left|\frac{\sin(n+1)\pi(x-x')}{(n+1)\sqrt{x-x'}}\right|\cdot\frac{dx'}{|\sqrt{x-x'}|}$$

$$< K\epsilon\int_0^1 \frac{dx'}{|\sqrt{x-x'}|} \qquad (n \geqq m;\ 0 \leqq x \leqq 1),$$

where K is a positive constant. But

$$(87)\qquad \int_0^1 \frac{dx'}{|\sqrt{x-x'}|} = \int_{-x}^{1-x} \frac{d\alpha}{\sqrt{|\alpha|}} < 2\int_0^1 \frac{d\alpha}{\sqrt{\alpha}} = 4.$$

Combining (87) with (86) we see that the integral in the second term in (84) approaches zero uniformly for all values of x in the interval (43) as n becomes infinite, and hence the second term itself approaches zero uniformly in the same interval.

The first term on the right hand side of (83) may be broken up into the sum of the following three integrals:

$$(88)\quad \begin{aligned} &-\frac{x}{4}\int_0^1 \frac{\sin[q(x+x')-2\alpha]}{\sin\pi\dfrac{x+x'}{2}}\sqrt{x'}f(x')\frac{\sin\pi(x+x')}{\sin\pi\dfrac{x+x'}{2}}dx', \\ &-\frac{x}{4n}\int_0^1 \frac{\sin[q(x+x')-2\alpha]}{\sin\pi\dfrac{x+x'}{2}}\sqrt{x'}f(x')\frac{\sin\pi(x+x')}{\sin\pi\dfrac{x+x'}{2}}dx', \\ &+\frac{x}{4}\int_0^1 \frac{\sin[q(x+x')-2\alpha]}{\sin\pi\dfrac{x+x'}{2}}\sqrt{x'}f(x')\frac{\sin(n+1)\pi(x+x')}{n\sin\pi\dfrac{x+x'}{2}}dx', \end{aligned}$$

provided that we can show that each of these integrals converges for all values of x in the interval (43). For this purpose we must consider the values of q.

By comparing equations (20), (44), and (22) we see that (1) may be written

$$\frac{A\,[\,8(\nu l+h)-l(4\nu^2+8\nu+3)]}{8\lambda}\cos(\lambda-\alpha)-Al\sin(\lambda-\alpha)+\frac{\psi(\lambda)}{\lambda^2}=0.$$

Consequently, except in the case $l=0$, the large roots of this equation coincide more and more nearly with the values

$$\alpha+k\pi,$$

where k takes on large positive integral values. In the exceptional case $l=0$ they coincide more and more nearly with

$$\frac{\pi}{2}+\alpha+k\pi.$$

Since, by (31), we see that λ_n coincides more and more nearly with $q+n\pi$, it follows that according as the l of equation (1) is or is not zero,

$$q=\frac{\pi}{2}+\alpha+k_1\pi \qquad \text{or} \qquad q=\alpha+k_2\pi,$$

where k_1 and k_2 are positive or negative integers or zero. Consequently the expression

$$\frac{\sin\,[\,q(x+x')-2\alpha\,]}{\sin\pi\dfrac{x+x'}{2}} \tag{89}$$

approaches a finite limit as $x+x'$ approaches the limit 2, and hence the integrals in (88) converge for all values of n and all values of x in the interval (43).

It now remains to be shown that the expression (88) approaches a limit uniformly for all values of x in the interval (43), as n becomes infinite. The first term of that expression is independent of n and hence need not be considered further. The integral in the second term has for its integrand the product of a function of x and x' that is continuous in both variables and a function of x' that is finite and integrable. It therefore defines a continuous function of x, and consequently the second term approaches zero uniformly for all values of x in the interval (43), as n becomes infinite.

The same is true of the third term of (88). For, since the expression

$$\frac{\sin(n+1)\pi(x+x')}{n\sin\pi\left(\dfrac{x+x'}{2}\right)}$$

remains finite for all positive integral values of n and all values of x and x', it

follows that if ϵ is given positive and arbitrarily small, we can choose η so small that

$$(90)\quad \left| \frac{x}{4} \int_{1-\eta}^{1} \frac{\sin\left[q(x+x')-2\alpha\right]}{\sin \pi \frac{x+x'}{2}} \sqrt{x'} f(x') \frac{\sin(n+1)\pi(x+x')}{n \sin \pi \frac{x+x'}{2}} dx' \right| < \frac{\epsilon}{2}$$

$$(n = 1, 2, 3, \cdots;\ 0 \leqq x \leqq 1).$$

We can then choose δ so small that

$$(91)\quad \left| \frac{x}{4} \int_{0}^{1-\eta} \frac{\sin\left[q(x+x')-2\alpha\right]}{\sin \pi \frac{x+x'}{2}} \sqrt{x'} f(x') \frac{\sin(n+1)\pi(x+x')}{n \sin \pi \frac{x+x'}{2}} dx' \right| < \frac{\epsilon}{2}$$

$$(n = 1, 2, 3, \cdots;\ 0 \leqq x < \delta).$$

Finally, η and δ being fixed, we can choose m so large that

$$(92)\quad \left| \frac{x}{4n} \int_{0}^{1-\eta} \frac{\sin\left[q(x+x')-2\alpha\right]}{\sin \pi \frac{x+x'}{2}} \sqrt{x'} f(x') \frac{\sin(n+1)\pi(x+x')}{\sin \pi \frac{x+x'}{2}} dx' \right| < \frac{\epsilon}{2}$$

$$(n \geqq m;\ \delta \leqq x \leqq 1).$$

Combining (92), (91), and (90) we see that the third term of (88) approaches zero uniformly for all values of x in the interval (43) as n becomes infinite.

We have now shown that as n becomes infinite each of the terms on the right hand side of (83) approaches a limit uniformly for all values of x in the interval (43) and therefore it follows that the series whose general term is (82), or the second and third terms of (81) multiplied by $x^{\frac{3}{2}}$, is uniformly summable in the interval (43). Hence, as we have pointed out above, our lemma is proved.

We are now prepared to prove the following theorem:

THEOREM II. *If in the interval* $0 \leqq x \leqq 1$ *the function* $f(x)$ *is finite and integrable, then the series* (35) *will be summable at every point of the interval* $0 < x < 1$ *at which* $f(x)$ *is continuous or has a finite jump; and will be uniformly summable in every closed interval lying in that interval which does not include a point of discontinuity of* $f(x)$.

Since $f(x)$ is finite and integrable,

$$\sqrt{x} f(x) \cos(qx - \alpha) \qquad \text{and} \qquad \sqrt{x} f(x) \sin(qx - \alpha)$$

are finite and integrable, and hence it follows from a theorem due to FEJÉR* that the two series whose general terms are the first and second terms of (41), respectively, are summable at every point of the interval $0 < x < 1$ at which the function $f(x)$ is continuous or has a finite jump, these series being essentially equivalent to the cosine development and sine development, respectively, of $\sqrt{x} f(x) \cos(qx - \alpha)$ and $\sqrt{x} f(x) \sin(qx - \alpha)$. We know from Lemma 5

* Loc. cit., p. 53.

that the series which has for its general term the third term of (41) is uniformly summable throughout the interval (42), and hence, since the c of that interval is arbitrary, is summable at every point of the interval $0 < x < 1$. Consequently the series (35), whose general term is the sum of the three terms occurring in (41), is summable at every point of the interval $0 < x < 1$ at which $f(x)$ is continuous or has a finite jump.

We know further from FEJÉR's work* that the series having for their general terms the first two terms in (41) are uniformly summable in every closed interval which lies in the interval $0 < x < 1$ and does not include a point of discontinuity of $f(x)$, and it follows from Lemma 5 that the series having for its general term the third term in (41) is uniformly summable in any such interval. The series (35) is therefore uniformly summable in any such interval and our theorem is completely established.

COROLLARY. *If we define* $S_n(x)$ *as in formula* (34), *we have for the series* (35)

$$|S_n(x)| < M \qquad (n = 1, 2, 3, \cdots; 0 < c \leqq x \leqq 1)$$

where M *is a positive constant.*

For each of the two series whose general terms are the first and second terms of (41) we have from FEJÉR's work†

$$|S_n(x)| < M_1 \qquad (n = 1, 2, 3, \cdots; 0 < c \leqq x \leqq 1), \tag{93}$$

where M_1 is a positive constant; for the series whose general term is the third term of (41) we have from Lemma 5

$$|S_n(x)| < M_2 \qquad (n = 1, 2, 3, \cdots; 0 < c \leqq x \leqq 1), \tag{94}$$

where M_2 is a positive constant. If we combine (93) and (94) the proof of our corollary follows at once.

Before proceeding to the proof of the next theorem, we wish to make the following convention. Let

$$\phi(x) = \cos^2(qx - \alpha) f(x) + \sin^2(qx - \alpha) f(x) + \sum_{n=1}^{\infty} v_n(x), \tag{95}$$

where v_n has the same meaning as in (41), and the series is to be evaluated as a summable series. The function $\phi(x)$ we will designate as the function *associated with* the series (35). It will be seen that $\phi(x)$ is thus defined at all points where $f(x)$ is defined, whereas the series (35) is not necessarily summable at points where $f(x)$ is discontinuous. At points where $f(x)$ is continuous, however, it is readily seen by a reference to Lemma 5 that the series (35) is summable to precisely the value $\phi(x)$.

* Loc. cit., p. 60.

† Loc. cit., p. 60.

THEOREM III. *If we multiply the series* (35) *by any function* $\chi(x)$ *that is finite and integrable in the interval* $0 \leqq x \leqq 1$, *and integrate the resulting series term by term from* δ * *to* 1, *we shall obtain a series which is summable to the value*

$$\int_{\delta}^{1} \phi(x)\chi(x)\,dx \tag{96}$$

where $\phi(x)$ *is the function associated with the series* (35).

Since the series whose general terms are the first and second terms of (41) differ from Fourier developments of finite and integrable functions only by factors which are continuous functions of x in the interval $\delta \leqq x \leqq 1$, we know from a theorem due to DE LA VALLÉE POUSSIN † that we can multiply each of them by any finite and integrable function $\chi(x)$ and integrate the resulting series term by term from δ to 1, and that the series thus obtained will converge to the value of the integral of the product of $\chi(x)$ and the function of which the series is a development multiplied by the function which appears as a factor of the series.

Since, from Lemma 5, the series whose general term is the third term of (41) is uniformly summable in the interval (42), we can multiply it by any finite and integrable function $\chi(x)$ and integrate it term by term from δ to 1, and the resulting series will be summable to the value of the integral of the product of $\chi(x)$ and the function to which the series is summable. Thus our theorem is proved.

§ 3. *Value of the development.*

The series (35) is usually given as the formal development of an arbitrary function in terms of Bessel functions, but there are cases in which this series, even if it is convergent or summable, will not have the desired value, an extra term being necessary in order that the development may represent the function developed.‡ However, in these cases the extra term appears naturally as part of the formal development if we keep in mind the method by which the formal development is commonly deduced.

In the problems of Mathematical Physics in which we wish to develop an arbitrary function in terms of certain normal functions, the latter functions appear as solutions of some ordinary differential equation. In the case in which the normal functions are Bessel functions the differential equation is usually

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + \left(\lambda^2 - \frac{\nu^2}{x^2}\right)u = 0 \tag{97}$$

* The quantity δ is any positive quantity less than 1.

† Cf. Annales de la Société Scientifique de Bruxelles, vol. 17 (1892-3). Cf. also BÔCHER'S article referred to above, p. 118, footnote.

‡ Cf. DINI: *Serie di Fourier*, and BRIDGMAN, Philosophical Magazine, vol. 16 (1908), p. 947.

where ν is a constant $\geqq 0$ and λ is a real constant (which may obviously be taken $\geqq 0$) to be determined in such a way that (97) has a solution, not identically zero, which remains finite when $x = 0$, and at the point $x = 1$ satisfies the condition

$$[lu'(x) + hu(x)]_{x=1} = 0 \tag{98}$$

where l and h are given constants not both zero.

If in (97) $\lambda \neq 0$, the most general solution which remains finite at $x = 0$ is

$$u = CJ_\nu(\lambda x), \tag{99}$$

and in order that this should satisfy (98) it is necessary and sufficient that

$$l\lambda J'_\nu(\lambda) + hJ_\nu(\lambda) = 0. \tag{100}$$

This is the equation for determining the values of λ *greater* than zero which we have to use. When, however, $\lambda = 0$, formula (99) no longer gives the general solution of (97) which remains finite when $x = 0$, this solution then being

$$u = Cx^\nu. \tag{101}$$

The latter solution satisfies (98) when and only when

$$l\nu + h = 0. \tag{102}$$

It is, therefore, only in this case that the value $\lambda = 0$ needs to be considered at all. In all other cases we develop our function in the form

$$f(x) = \sum_{n=1}^{\infty} A_n J_\nu(\lambda_n x),$$

where $\lambda_1, \lambda_2, \cdots$ are the successive *positive* roots of (100), and the coefficients A are to be determined by the ordinary formulas.

In the exceptional case (102) the λ's *greater* than zero are still determined by (100), which now reduces to

$$-lJ_{\nu+1}(\lambda) = 0.^* \tag{103}$$

But we also have to use the value $\lambda = 0$, and thus we are led to the development

$$f(x) = A_1 x^\nu + \sum_{n=2}^{\infty} A_n J_\nu(\lambda_n x), \tag{104}$$

where $\lambda_2, \lambda_3, \cdots$ are the positive roots of (103). Here not only are $\sqrt{x}J_\nu(\lambda_i x)$ and $\sqrt{x}J_\nu(\lambda_j x)$ orthogonal to each other, i. e.,

$$\int_0^1 xJ_\nu(\lambda_i x)J_\nu(\lambda_j x)\,dx = 0,$$

* Cf. for instance BYERLY, loc. cit., p. 223, formula (6).

but $x^{\nu+1}$ is also orthogonal to each of the functions $\sqrt{x}J_\nu(\lambda_n x)$, i. e.,

$$\int_0^1 x^{\nu+1} J_\nu(\lambda_n x)\,dx = 0.$$

Accordingly here also the coefficients can be determined in the ordinary way. For instance, to get A_1 we multiply (104) through by $x^{\nu+1}$ and integrate term by term from 0 to 1, thus obtaining *

$$\int_0^1 x^{\nu+1} f(x)\,dx = A_1 \int_0^1 x^{2\nu+1}\,dx = \frac{A_1}{2\nu+2},$$

so that

$$A_1 = (2\nu+2)\int_0^1 x^{\nu+1} f(x)\,dx.$$

If then, in developing our arbitrary function, we use *all* of the normal functions that satisfy (97) and (98) we shall naturally have an extra term in the case in which (102) is satisfied. Hence, what appears to be an exceptional case is really not exceptional at all, save in regard to the *notation* ordinarily used for expressing the solutions of (97) in terms of Bessel functions. Moreover, it is possible to adopt a notation which avoids the awkwardness of making a special case of the solution (101). For, if we set

$$F_\nu(\lambda, x) = \left(\frac{x}{2}\right)^\nu \sum_{i=0}^{\infty} \frac{(-1)^i(\lambda x)^{2i}}{2^{2i}\, i!\, \Gamma(\nu+i+1)} \tag{105}$$

so that

$$J_\nu(\lambda x) \equiv \lambda^\nu F_\nu(\lambda, x), \tag{105a}$$

then

$$u = CF_\nu(\lambda, x)$$

will be the most general solution of (97) which remains finite when $x = 0$, not only when $\lambda > 0$ but also when $\lambda = 0$, since in this last case it reduces to the form (101).

We now consider the series

$$\sum_{n=1}^{\infty} F_\nu(\lambda_n, x) \frac{\int_0^1 xf(x)F_\nu(\lambda_n, x)\,dx}{\int_0^1 x[F_\nu(\lambda_n, x)]^2\,dx} = \sum_{n=1}^{\infty} A_n F_\nu(\lambda_n, x), \tag{106}$$

where $\lambda_1, \lambda_2, \lambda_3, \cdots$ are the roots, positive or zero, of the equation

$$\left[l\frac{\partial}{\partial x}F_\nu(\lambda_1 x) + hF_\nu(\lambda_1 x)\right]_{x=1} = 0 \dagger \tag{107}$$

* It is understood of course that we are using here merely the formal device ordinarily employed for finding the coefficients of the development without entering into the question of whether or not we have a right to integrate the series term by term.

† The *positive* roots of this equation are obviously identical with the *positive* roots of equation (1), and hence all that we have proved for the latter roots holds for the former ones.

arranged in increasing order of magnitude. The equation (107) will have a zero root when and only when (102) is fulfilled, and thus the series (106) will give the formal development of the arbitrary function $f(x)$ for all cases. This series will be identical with (35) except when (102) is fulfilled, and in that case will differ from it only in having an extra term. Since, however, this term will be finite and continuous it will not affect the summability or uniform summability of (35) and all that we have proved in the previous section with regard to that series will hold in all essential respects with regard to the series (106).

Thus if we write the general term of (106) in the form

$$(108)\quad \begin{aligned} &\frac{2}{\sqrt{x}}\cos(qx-\alpha)\cos n\pi x\int_0^1\sqrt{x}\,f(x)\cos(qx-\alpha)\cos n\pi x\,dx \\ &+\frac{2}{\sqrt{x}}\sin(qx-\alpha)\sin n\pi x\int_0^1\sqrt{x}f(x)\sin(qx-\alpha)\sin n\pi x\,dx+w_n(x), \end{aligned}$$

the series

$$\sum_{n=1}^{\infty} w_n(x)$$

will be uniformly summable in the interval (42), and if we define the function

$$(109)\quad \phi(x)=\cos^2(qx-\alpha)f(x)+\sin^2(qx-\alpha)f(x)+\sum_{n=1}^{\infty}w_n(x)$$

to be the function *associated with* the series (106), Theorem III holds without further change if we replace the series (35) by the series (106) in the statement of that theorem. Moreover $\phi(x)$ will have the value to which the series (106) is summable at every point at which $f(x)$ is continuous.

We now wish to show that the series (106) is summable to the value

$$(110)\qquad \tfrac{1}{2}[f(x+0)+f(x-0)]$$

at every point of the interval $0<x<1$ at which $f(x)$ is continuous or at which it has a finite jump, provided the latter points lie within an interval at every other point of which $f(x)$ is continuous. We will begin by proving some lemmas.

LEMMA 6. *If $f(x)$ is finite and integrable and $\phi(x)$ is the function associated with the series* (106), *we have*

$$(111)\qquad \int_0^1 x\phi(x)F_\nu(\lambda_k,x)\,dx=\int_0^1 xf(x)F_\nu(\lambda_k,x)\,dx\quad(k=1,2,3,\cdots),$$

where $\lambda_1,\lambda_2,\lambda_3,\cdots$ are the roots, positive or zero, of equation (107), *arranged in increasing order of magnitude, and $F_\nu(\lambda,x)$ is defined by equation* (105).

We have, from the modification of Theorem III just referred to, the following equality*

$$\int_\delta^1 x\phi(x)F_\nu(\lambda_k, x)dx = A_1\int_\delta^1 xF_\nu(\lambda_1, x)F_\nu(\lambda_k, x)dx + A_2\int_\delta^1 xF_\nu(\lambda_2, x)F_\nu(\lambda_k, x)dx + \cdots. \tag{112}$$

Moreover $F_\nu(\lambda_n, x)$ and $F_\nu(\lambda_k, x)$ satisfy differential equations of the form (97). From these we readily deduce, if we use accents to denote differentiation with regard to x,

$$\begin{aligned}(\lambda_n^2 - \lambda_k^2)xF_\nu(\lambda_n, x)F_\nu(\lambda_k, x) &= x[F_\nu(\lambda_n, x)F_\nu''(\lambda_k, x) - F_\nu(\lambda_k, x)F_\nu''(\lambda_n, x)] \\ &\quad + [F_\nu(\lambda_n, x)F_\nu'(\lambda_k, x) - F_\nu(\lambda_k, x)F_\nu'(\lambda_n, x)] \\ &= \frac{d}{dx}x[F_\nu(\lambda_n, x)F_\nu'(\lambda_k, x) - F_\nu(\lambda_k, x)F_\nu'(\lambda_n, x)] \\ &= \frac{d}{dx}x[\lambda_n^2 F_\nu(\lambda_k, x)F_{\nu+1}(\lambda_n, x) - \lambda_k^2 F_\nu(\lambda_n, x)F_{\nu+1}(\lambda_k, x)],\dagger\end{aligned}$$

and consequently when $\lambda_n \neq \lambda_k$

$$\int_\delta^1 xF_\nu(\lambda_n, x)F_\nu(\lambda_k, x)dx = \frac{1}{\lambda_n^2 - \lambda_k^2}[x\{\lambda_n^2 F_\nu(\lambda_k, x)F_{\nu+1}(\lambda_n, x) - \lambda_k^2 F_\nu(\lambda_n, x)F_{\nu+1}(\lambda_k, x)\}]_\delta^1. \tag{113}$$

It follows without difficulty from equations (105a) and (44) that at the upper limit the quantity in square brackets on the right hand side of (113) may be written

$$\frac{\psi_1(\lambda_n)}{\lambda_n^{\nu-\frac{1}{2}}}.$$

At the lower limit it may be written

$$\frac{\psi_1(\lambda_n)}{\lambda_n^{\nu+\frac{1}{2}}},$$

if δ lies in the interval

$$0 < \delta < \frac{c}{\lambda_n},$$

* It is understood of course that the series on the right hand side of (112) is to be evaluated as a summable series, as we have yet no reason to suppose that it is convergent.

† This last reduction follows from the formula

$$F_\nu'(\lambda, x) = -\lambda^2 F_{\nu+1}(\lambda, x) + \frac{\nu}{x}F_\nu(\lambda, x).$$

Cf. BYERLY, loc. cit., p. 223, formula (6).

where c is any positive constant <1, since we have from $(105a)$

$$F_\nu(\lambda_n, x) = \frac{J_\nu(\lambda_n x)}{\lambda_n^\nu} = \frac{\psi_1(\lambda_n)}{\lambda_n^\nu}, \qquad F_{\nu+1}(\lambda_n, x) = \frac{\psi_1(\lambda_n)}{\lambda_n^{\nu+1}}.$$

If $\delta > c/\lambda_n$ it may be written

$$\frac{\psi_1(\lambda_n)}{\lambda_n^{\nu-\frac{1}{2}}},$$

since for such values of δ we have from $(105a)$ and the asymptotic values of $J_\nu(x)$ and $J_{\nu+1}(x)$

$$F_\nu(\lambda_n, \delta) = \frac{\psi_1(\lambda_n)}{\sqrt{\lambda_n \delta}\lambda_n^\nu} + \frac{\psi_1(\lambda_n)}{(\lambda_n\delta)^{\frac{3}{2}}\lambda_n^\nu} = \frac{1}{\sqrt{\delta}}\left[\frac{\psi_1(\lambda_n)}{\lambda_n^{\nu+\frac{1}{2}}} + \frac{\psi_1(\lambda_n)}{(\lambda_n\delta)\lambda_n^{\nu+\frac{1}{2}}}\right] = \frac{\psi_1(\lambda_n)}{\sqrt{\delta}\,\lambda_n^{\nu+\frac{1}{2}}},$$

$$F_{\nu+1}(\lambda_n, \delta) = \frac{\psi_1(\lambda_n)}{\sqrt{\delta}\,\lambda_n^{\nu+\frac{3}{2}}}.$$

Finally if $\delta = 0$ the expression in square brackets on the right hand side of (113) vanishes at the lower limit. Hence we have

$$(114) \qquad \left|\int_\delta^1 x F_\nu(\lambda_n, x) F_\nu(\lambda_k, x)\, dx\right| < \frac{M}{\lambda_n^{\nu-\frac{1}{2}} \cdot |\lambda_n^2 - \lambda_k^2|} \qquad (n \neq k,\ 0 \leqq \delta < 1),$$

where M is a positive constant. Furthermore it follows from (46), (50), and $(105a)$ that

$$(115) \qquad A_n = \frac{2\lambda_n^{\nu+\frac{1}{2}}}{A}\int_0^1 \sqrt{x} f(x) \cos(\lambda_n x - \alpha)\, dx + \lambda_n^{\nu-\frac{1}{2}}\psi_1(\lambda_n).$$

Consequently, in view of (114) and (115), we have for the general term of the series on the right hand side of (112)

$$\left|A_n \int_\delta^1 x F_\nu(\lambda_n, x) F_\nu(\lambda_k, x)\, dx\right| < \frac{2M\lambda_n}{A|\lambda_n^2 - \lambda_k^2|}\left|\int_0^1 \sqrt{x} f(x) \cos(\lambda_n x - \alpha)\, dx\right| + \frac{M|\psi_1(\lambda_n)|}{|\lambda_n^2 - \lambda_k^2|} \qquad (\delta \geqq 0).$$

The second term on the right hand side of the above inequality is, in view of the first inequality in (23), the general term of a convergent series, and it follows from Lemma 4 that the same is true for the first term. Hence the series on the right hand side of (112) converges uniformly for all values of δ in the interval $0 \leqq \delta < 1$, and moreover it converges to the value of the integral on the left hand side for all values of $\delta > 0$, since it is summable to that value. Since the convergence is uniform, we may let the δ of equation (112) approach zero. But in view of the fact that for the different values of λ_k that enter, the functions $\sqrt{x}F_\nu(\lambda_k, x)$ are orthogonal to each other, all the terms but one on the right hand side of (112) drop out, and our lemma is established.

LEMMA 7. *The function $\phi(x)$ associated with the series* (106) *does not become infinite to an order higher than $x^{-\frac{1}{2}}$ as we approach the point $x=0$.*

By definition the function $\phi(x)$ is equal to the sum of two functions that are finite and integrable in the interval (43), plus the series

$$\sum_{n=1}^{\infty} v_n(x). \tag{116}$$

By Lemma 5 the series (116) is equal to the product of $x^{-\frac{1}{2}}$ into the series

$$\sum_{n=1}^{\infty} \bar{v}_n(x).$$

But by the same lemma this latter series is uniformly summable and its terms are continuous in the interval (43). Hence it defines a continuous function in this interval, and consequently the series (116) defines a function that does not become infinite to an order higher than $x^{-\frac{1}{2}}$ when we approach the point $x=0$.

Our lemma is therefore proved.

THEOREM IV. *If we have a function $\chi(x)$ which in the interval $0 \leqq x \leqq 1$ is finite everywhere save in the neighborhood of the point $x=0$, and if the integral*

$$\int_0^1 x\chi(x)\,dx$$

converges absolutely, and if furthermore

$$\int_0^1 xF_\nu(\lambda_n, x)\chi(x)\,dx = 0 \qquad (n=1, 2, 3, \cdots) \tag{117}$$

where $\lambda_1, \lambda_2, \lambda_3, \cdots$, are the roots, positive or zero, of equation (107) *arranged in increasing order of magnitude, then $\chi(x)$ is zero at every point at which it is continuous.*

We know from some theorems due to DINI * that we can develop each of the functions

$$x^{\frac{1}{2}} \sin k\pi x \qquad (k=1, 2, 3, \cdots) \tag{118}$$

in a series of the form

$$a_1F_\nu(\lambda_1, x) + a_2F_\nu(\lambda_2, x) + a_3F_\nu(\lambda_3, x) + \cdots \dagger \tag{119}$$

and that the series will converge to the value of the function developed at every point of the interval $0 < x < 1$. We are going to show further that for the functions we are considering the developments will converge *uniformly* throughout the interval $0 \leqq x \leqq 1$.

* *Serie di Fourier*, vol. 1, pp. 246–269. DINI treats a very general class of functions of which the functions (118) are a special case.

† There is an error in DINI's formulæ for the special case in which $\lambda_1 = 0$; he omits the factor $F_\nu(\lambda_1, x)$ from the first term of the series.

We will consider first the coefficient of the general term of the series (119) and for simplicity write λ for λ_n. We have then

$$a_n = \frac{\int_0^1 x^2 \sin k\pi x F_\nu(\lambda, x)\,dx}{\int_0^1 x[F_\nu(\lambda, x)]^2 dx}. \tag{120}$$

It follows from equation (46) and equation (105a) that

$$\int_0^1 x[F_\nu(\lambda, x)]^2 dx = \frac{A^2}{2\lambda^{2\nu+1}} + \frac{\psi_1(\lambda)}{\lambda^{2\nu+2}}.$$

We may therefore choose λ so large that

$$|a_n| < \frac{4\lambda^{2\nu+1}}{A^2}\left|\int_0^1 x^2 \sin k\pi x F_\nu(\lambda, x)\,dx\right|. \tag{121}$$

But we have

$$\int_0^1 x^2 \sin k\pi x F_\nu(\lambda, x)\,dx = \int_0^{\frac{1}{\lambda}} + \int_{\frac{1}{\lambda}}^1 \tag{122}$$

and if we set $\lambda x = y$

$$\begin{aligned}\int_0^{\frac{1}{\lambda}} x^2 \sin k\pi x F_\nu(\lambda, x)\,dx &= \frac{1}{\lambda^\nu}\int_0^{\frac{1}{\lambda}} x^2 \sin k\pi x J_\nu(\lambda x)\,dx \\ &= \frac{1}{\lambda^{\nu+3}}\int_0^1 y^2 \sin\frac{k\pi y}{\lambda} J_\nu(y)\,dy.\end{aligned} \tag{123}$$

Moreover we have from equations (49) and (105a)

$$\begin{aligned}&\int_{\frac{1}{\lambda}}^1 x^2 \sin k\pi x F_\nu(\lambda, x)\,dx \\ &= \frac{A}{\lambda^{\nu+\frac{1}{2}}}\int_{\frac{1}{\lambda}}^1 x^{\frac{3}{2}} \sin k\pi x \cos(\lambda x - \alpha)dx + \frac{A'}{\lambda^{\nu+\frac{3}{2}}}\int_{\frac{1}{\lambda}}^1 x^{\frac{1}{2}} \sin k\pi x \sin(\lambda x - \alpha)dx + \frac{\psi_1(\lambda)}{\lambda^{\nu+\frac{5}{2}}} \\ &= \frac{A}{\lambda^{\nu+\frac{1}{2}}}\int_0^1 x^{\frac{3}{2}} \sin k\pi x \cos(\lambda x - \alpha)dx + \frac{A'}{\lambda^{\nu+\frac{3}{2}}}\int_0^1 x^{\frac{1}{2}} \sin k\pi x \sin(\lambda x - \alpha)dx \\ &\quad - \frac{A}{\lambda^{\nu+\frac{1}{2}}}\int_0^{\frac{1}{\lambda}} x^{\frac{3}{2}} \sin k\pi x \cos(\lambda x - \alpha)dx - \frac{A'}{\lambda^{\nu+\frac{3}{2}}}\int_0^{\frac{1}{\lambda}} x^{\frac{1}{2}} \sin k\pi x \sin(\lambda x - \alpha)dx + \frac{\psi_1(\lambda)}{\lambda^{\nu+\frac{5}{2}}}.\end{aligned} \tag{124}$$

But we have from equation (57) for $\lambda = \lambda_n$ *

$$\cos(\lambda x - \alpha) = \cos\left(n\pi x + qx - \alpha + \frac{x\psi_1(\lambda_n)}{n}\right)$$

* In the special case in which $\lambda_1 = 0$ we must take $\lambda = \lambda_{n+1}$ for (125) to hold true. However, since we are dealing here only with a question of convergence, this is unessential.

$$
\begin{aligned}
&= \cos(n\pi x + qx - \alpha)\cos\frac{x\psi_1(\lambda_n)}{n} - \sin(n\pi x + qx - \alpha)\sin\frac{x\psi_1(\lambda_n)}{n} \\
&= \cos(n\pi x + qx - \alpha) + [\cos(n\pi x + qx - \alpha)]\cdot\left[\cos\frac{x\psi_1(\lambda_n)}{n} - 1\right] \\
&\qquad - \sin(n\pi x + qx - \alpha)\sin\frac{x\psi_1(\lambda_n)}{n}.
\end{aligned} \tag{125}
$$

Moreover

$$
\begin{aligned}
\int_0^1 x^{\frac{1}{2}} \sin k\pi x \cos(n\pi x + qx - \alpha)\,dx &= \int_0^1 x^{\frac{1}{2}} \sin k\pi x \cos(qx - \alpha)\cos n\pi x\,dx \\
&\quad - \int_0^1 x^{\frac{1}{2}} \sin k\pi x \sin(qx - \alpha)\sin n\pi x\,dx = \frac{\psi_1(n)}{n^2},
\end{aligned} \tag{126}
$$

and from an integration by parts

$$
\begin{aligned}
&\int_0^1 x^{\frac{1}{2}} \sin k\pi x \sin(n\pi x + qx - \alpha)\sin\frac{x\psi_1(\lambda_n)}{n}\,dx \\
&= \frac{1}{n\pi + q}\left[-x^{\frac{1}{2}} \sin k\pi x \cos(n\pi x + qx - \alpha)\sin\frac{x\psi_1(\lambda_n)}{n}\right]_0^1 \\
&\quad + \frac{3}{2(n\pi + q)}\int_0^1 x^{\frac{1}{2}} \sin k\pi x \cos(n\pi x + qx - \alpha)\sin\frac{x\psi_1(\lambda_n)}{n}\,dx \\
&\quad + \frac{k\pi}{n\pi + q}\int_0^1 x^{\frac{1}{2}} \cos k\pi x \cos(n\pi x + qx - \alpha)\sin\frac{x\psi_1(\lambda_n)}{n}\,dx \\
&\quad + \frac{\psi_1(\lambda_n)}{n(n\pi + q)}\int_0^1 x^{\frac{1}{2}} \sin k\pi x \cos(n\pi x + qx - \alpha)\cos\frac{x\psi_1(\lambda_n)}{n}\,dx.
\end{aligned} \tag{127}
$$

Consequently, since

$$
\sin\frac{x\psi_1(\lambda_n)}{n} = \frac{\psi(x, \lambda_n)}{n}, \qquad \cos\frac{x\psi_1(\lambda_n)}{n} - 1 = \frac{\psi(x, \lambda_n)}{n^2},
$$

it follows from (125), (126), and (127) that

$$
\int_0^1 x^{\frac{1}{2}} \sin k\pi x \cos(\lambda x - \alpha)\,dx = \frac{\psi_1(n)}{n^2}. \tag{128}
$$

In a similar fashion it may be shown that

$$
\int_0^1 x^{\frac{1}{2}} \sin k\pi x \sin(\lambda x - \alpha)\,dx = \frac{\psi_1(n)}{n}, \tag{129}
$$

and consequently, since from Theorem I

$$
\lambda_n > Cn, \tag{130}
$$

* Cf. PICARD, *Traité d'Analyse*, 2d ed., vol. 1, p. 256.

we have from (124), (128) and (129)

$$\int_{\frac{1}{\lambda}}^{1} x^2 \sin k\pi x F_\nu(\lambda, x)\,dx = \frac{\psi_1(n)}{n^{\nu+\frac{3}{2}}}. \tag{131}$$

Combining (122), (123), (130), and (131) we obtain

$$\int_{0}^{1} x^2 \sin k\pi x F_\nu(\lambda, x)\,dx = \frac{\psi_1(n)}{n^{\nu+\frac{3}{2}}}. \tag{132}$$

Since we know that the interval between two successive positive roots of equation (107) approaches π as a limit, it follows without difficulty that a positive constant M can be determined such that

$$\lambda_n < Mn. \tag{133}$$

Combining (121), (132), and (133) we get

$$|a_n| < \frac{K_1 \lambda_n^\nu}{n^{\frac{3}{2}}}, \tag{134}$$

where K_1 is a positive constant. But since the Bessel function of any order has a maximum absolute value for all real positive values of the argument, we have from (105a)

$$|F_\nu(\lambda_n, x)| < \frac{K_2}{\lambda_n^\nu}, \tag{135}$$

where K_2 is a positive constant. Combining (134) and (135) we have

$$|a_n F_\nu(\lambda_n, x)| < \frac{K}{n^{\frac{3}{2}}},$$

K being a positive constant. Hence the series that we are discussing converge absolutely and uniformly in the interval $0 \leqq x \leqq 1$.

Since the developments of the functions (118) in series of the form (119) converge uniformly in the interval $0 \leqq x \leqq 1$, we have a right to multiply them by $x\chi(x)$ and integrate them term by term from 0 to 1. Performing this operation we obtain in view of (117)

$$\int_0^1 x^2 \sin k\pi x \chi(x)\,dx = 0 \qquad (k = 1, 2, 3, \cdots).$$

Consequently, since any finite and integrable function whose Fourier's constants all vanish is zero at every point at which it is continuous,* it follows that $x^2\chi(x)$ and hence $\chi(x)$ is zero at every point of the interval $0 \leqq x \leqq 1$ at which it is continuous.

*Cf. HURWITZ, Mathematische Annalen, vol. 57 (1903), p. 440.

THEOREM V. *If $f(x)$ satisfies the conditions of Theorem II and if $\lambda_1, \lambda_2, \lambda_3, \cdots$ are the roots, positive or zero, of equation* (107) *arranged in order of increasing magnitude, then the series* (106) *will be summable to the value* (110) *at every point of the interval $0 < x < 1$ at which $f(x)$ is continuous or at which it has a finite jump, provided the latter points lie in an interval at every other point of which $f(x)$ is continuous; and will be uniformly summable to the value $f(x)$ in every closed interval lying in the interval $0 < x < 1$ which does not include a point of discontinuity of $f(x)$.*

We have from equation (111)

$$\int_0^1 x\,[\phi(x) - f(x)]\,F_\nu(\lambda_k, x)\,dx = 0 \qquad (k = 1, 2, 3, \cdots) \tag{136}$$

where $\phi(x)$ is a function that has the value of the series (106) at every point at which $f(x)$ is continuous (see page 421).

By hypothesis $f(x)$ is finite and integrable in the interval $0 \leqq x \leqq 1$. We know from the definition of $\phi(x)$ that $\phi(x)$ is finite and integrable in the interval $0 < \delta \leqq x \leqq 1$ where δ may be taken as small as we please. Moreover, it follows from Lemma 7 that $\phi(x)$ cannot become infinite at the point $x = 0$ to an order higher than $x^{-\frac{1}{2}}$. Hence $\phi(x)$ is finite in the interval $0 \leqq x \leqq 1$ except in the neighborhood of the point $x = 0$, and $x\phi(x)$ is absolutely integrable from 0 to 1.

Consequently, in view of Theorem IV, it follows from equations (136) that

$$\chi(x) = \phi(x) - f(x) = 0 \tag{137}$$

at every point of the interval $0 \leqq x \leqq 1$ at which it is continuous. Now we know from the definition of $\phi(x)$ that $\phi(x)$ is continuous at every point of the interval $0 < x < 1$ at which $f(x)$ is continuous. Hence it follows from equation (137) that at every point of the interval $0 < x < 1$ at which $f(x)$ is continuous the series (106) is summable to the value (110).*

We consider next the points at which $f(x)$ has a finite jump and which lie in an interval at every other point of which $f(x)$ is continuous. We know that at every interior point of that interval, except the point at which $f(x)$ has a finite jump, the series (106) is summable to $f(x)$. Moreover, that series is the sum of a series that is uniformly summable and continuous in the neighborhood of

* Since $f(x) = \phi(x)$ at every point at which $f(x)$ is continuous and since from (109)

$$\varphi(x) = f(x) + \sum_{n=1}^{\infty} w_n(x)$$

it follows that

$$\sum_{n=1}^{\infty} w_n(x) \tag{a}$$

is zero at every point at which $f(x)$ is continuous. However, since the series (a) defines a function that is continuous throughout the interval $0 < x \leqq 1$, it follows that it defines a function that is identically zero in that interval, so that $\phi(x) \equiv f(x)$.

that point plus two series that are essentially Fourier developments of functions having a finite jump at that point. Hence the finite jump in the function defined by the sum of the two Fourier developments has the same magnitude as the finite jump in $f(x)$. However, it follows from FEJÉR's work that the Fourier developments are summable at a point at which the function they define has a finite jump, to a value half way between the limiting values of the function. Consequently the series (106) is summable to the value (110) at the points which we are discussing.

That the series is uniformly summable to the value $f(x)$ in every closed interval lying in the interval $0 < x < 1$ which does not include a point of discontinuity of $f(x)$, is an immediate consequence of Theorem II and the fact that the series is summable to the value $f(x)$ at every point of such an interval.

§ 4. *Convergence factors and applications.*

THEOREM VI. *If the series*

$$u_1(x) + u_2(x) + u_3(x) + \cdots \tag{138}$$

is uniformly summable in the interval $a \leqq x \leqq b$ and has the value $f(x)$, then the series

$$u_1(x)e^{-\lambda_1\alpha} + u_2(x)e^{-\lambda_2\alpha} + u_3(x)e^{-\lambda_3\alpha} + \cdots, \tag{139}$$

where $\lambda_1, \lambda_2, \lambda_3, \cdots$ are the successive positive roots of (1), *will be absolutely convergent in the region*

$$\begin{pmatrix} \alpha > 0 \\ a \leqq x \leqq b \end{pmatrix} \tag{140}$$

and its value $u(\alpha, x)$ will approach $f(x)$ uniformly for all values of x in the interval $a \leqq x \leqq b$, as α approaches $+0$.

We will first show that the convergence factors

$$e^{-\lambda_1\alpha},\ e^{-\lambda_2\alpha},\ e^{-\lambda_3\alpha}, \cdots \tag{141}$$

satisfy the conditions of a general theorem due to BROMWICH.* It will then follow that the series (139) is absolutely convergent in the region (140) and that its value approaches $f(x)$ for all values of x in the interval $a \leqq x \leqq b$ as α approaches $+0$. That this approach is uniform can easily be seen by a slight modification of the reasoning in BROMWICH's theorem.

BROMWICH's conditions are

$$\left.\begin{array}{ll} (\alpha) & \sum_{n=\mu}^{n=\nu} n|\Delta^2 v_n| < K \\ (\beta) & \lim_{n=\infty} nv_n = 0 \end{array}\right\} \alpha > 0,$$

$$(\gamma) \qquad \lim_{\alpha=+0} v_n = 1,$$

*Cf. Mathematische Annalen, vol. 65 (1908), p. 351.

where μ and ν are any two integers, $v_1, v_2, v_3, \cdots$, are the convergence factors, $\Delta^2 v_n$ represents their second differences, and K is a positive constant. That the functions (141) satisfy (β) and (γ) is at once obvious. It only remains to examine (α).

In order to establish this latter condition we will prove that the second differences of the functions (141) satisfy the inequalities *

$$|\Delta^2 e^{-\lambda_n a}| < N\alpha^2 + \frac{M\alpha}{n^2} \qquad (0 < na \leqq c;\ n = 1, 2, 3, \cdots), \tag{142}$$

$$|\Delta^2 e^{-\lambda_n a}| < \frac{L}{n^3 \alpha} \qquad (a > 0;\ n = 1, 2, 3, \cdots). \tag{143}$$

It will then follow without difficulty that (α) holds.†

Let

$$\lambda_{n+1} - \lambda_n = \gamma_n, \qquad \lambda_{n+2} - \lambda_{n+1} = \gamma_{n+1}. \tag{144}$$

Then by Theorem I

$$|\gamma_n - \pi| < \frac{K}{n^2}, \qquad |\gamma_{n+1} - \pi| < \frac{K}{(n+1)^2} < \frac{K}{n^2},$$

and hence

$$|\gamma_{n+1} - \gamma_n| = |(\gamma_{n+1} - \pi) - (\gamma_n - \pi)| \leqq |\gamma_{n+1} - \pi| + |\gamma_n - \pi| < \frac{2K}{n^2}. \tag{145}$$

But we have

$$e^{-\lambda_n a} - 2e^{-(\lambda_n + \gamma_n)a} + e^{-(\lambda_n + 2\gamma_n)a} = e^{-(\lambda_n + \theta_1 \gamma_n)a} \gamma_n^2 \alpha^2 \qquad (0 < \theta_1 < 2), \tag{146}$$

and since, in view of (144), γ_n has an upper limit, we obtain

$$e^{-\lambda_n a} - 2e^{-(\lambda_n + \gamma_n)a} + e^{-(\lambda_n + 2\gamma_n)a} < N\alpha^2, \tag{147}$$

where N is a positive constant.

We have furthermore from the Law of the Mean

$$e^{-\lambda_{n+2} a} - e^{-(\lambda_n + 2\gamma_n)a} = -e^{-[\lambda_n + 2\gamma_n + \theta_2(\gamma_{n+1} - \gamma_n)a]} (\gamma_{n+1} - \gamma_n)\alpha \qquad (|\theta_2| < 1), \tag{148}$$

from which we obtain with the aid of (145)

$$|e^{-\lambda_{n+2} a} - e^{-(\lambda_n + 2\gamma_n)a}| < \frac{2K\alpha}{n^2} = \frac{M\alpha}{n^2}, \tag{149}$$

where M is a positive constant.

* It may be noted that the inequalities (142) and (143) are analogous to conditions (d) and (e) in a theorem on convergence factors proved by the writer (these Transactions, vol. 8 (1907), p. 300), which is a special case of BROMWICH's theorem. The necessity for distinguishing between the differences where $na \leqq c$ and $na > c$ often arises in the consideration of convergence factors that occur in practice.

† The proof of this fact in the present case is analogous to the proof in the case where conditions (d) and (e) of the theorem mentioned in the previous footnote hold. This latter proof is given on pp. 353 and 354 of BROMWICH's article referred to above.

Finally from (147) and (149) we have

$$|e^{-\lambda_n\alpha} - 2e^{-\lambda_{n+1}\alpha} + e^{-\lambda_{n+2}\alpha}| \leqq |e^{-\lambda_n\alpha} - 2e^{-(\lambda_n+\gamma_n)\alpha} + e^{-(\lambda_n+2\gamma_n)\alpha}|$$
$$+ |e^{-\lambda_{n+2}\alpha} - e^{-(\lambda_n+2\gamma_n)\alpha}| < N\alpha^2 + \frac{M\alpha}{n^2},$$

and hence the inequality (142) is satisfied.

We obtain from (146), (148), and (145)

$$|e^{-\lambda_n\alpha} - 2e^{-\lambda_{n+1}\alpha} + e^{-\lambda_{n+2}\alpha}| \leqq e^{-(\lambda_n+\theta_1\gamma_n)\alpha}\gamma_n^2\alpha^2 + |e^{-[\lambda_n+2\gamma_n+\theta_2(\gamma_{n+1}-\gamma_n)]\alpha}(\gamma_{n+1}-\gamma_n)\alpha|$$
$$< \frac{2\gamma_n^2\alpha^2}{(\lambda_n+\theta_1\gamma_n)^3\alpha^3} + \frac{2K\alpha}{n^2[\lambda_n+2\gamma_n+\theta_2(\gamma_{n+1}-\gamma_n)]^2\alpha^2} < \frac{2\gamma_n^2}{\lambda_n^3\alpha} + \frac{2K}{n^2\lambda_n^2\alpha}$$
$$< \frac{\frac{2\gamma_n^2}{C^3}}{n^3\alpha} + \frac{\frac{2K}{C^2}}{n^4\alpha} < \frac{L}{n^3\alpha} \quad (n=1, 2, 3, \cdots; \alpha>0),$$

where L is a positive constant, and therefore the inequality (143) holds.

Consequently, as we have pointed out above, our theorem is proved.

An analogous theorem about slightly different convergence factors is the following:

THEOREM VII. *If the series* (138) *is uniformly summable in the interval* $a \leqq x \leqq b$ *and has the value* $f(x)$, *then the series*

$$u_1(x)e^{-\lambda_1^2\alpha^2} + u_2(x)e^{-\lambda_2^2\alpha^2} + u_3(x)e^{-\lambda_3^2\alpha^2} + \cdots, \tag{150}$$

where $\lambda_1, \lambda_2, \lambda_3, \cdots$ *are the successive positive roots of* (1), *will be absolutely convergent in the region* (140) *and its value* $u(\alpha, x)$ *will approach* $f(x)$ *uniformly for all values of* x *in the interval* $a \leqq x \leqq b$, *as* α *approaches* $+0$.

This theorem also can be proved by showing that the convergence factors

$$e^{-\lambda_1^2\alpha^2},\ e^{-\lambda_2^2\alpha^2},\ e^{-\lambda_3^2\alpha^2},\ \cdots \tag{151}$$

satisfy the conditions of BROMWICH's theorem. The proof is quite analogous to the previous proof, save that instead of the inequalities (142) and (143) we establish the slightly different inequalities

$$|\Delta^2 e^{-\lambda_n^2\alpha^2}| < N\alpha^2 \qquad (0 < n\alpha \leqq c;\ n=1, 2, 3, \cdots), \tag{152}$$

$$|\Delta^2 e^{-\lambda_n^2\alpha^2}| < \frac{L}{n^4\alpha^2} \qquad (\alpha>0;\ n=1, 2, 3, \cdots). \tag{153}$$

The results just obtained have a direct application to many problems in mathematical physics.

Problem I. The problem of determining the flow of heat in an infinite, circular cylinder, when the initial and boundary conditions depend only on the

perpendicular distance from the axis of the cylinder, reduces to the determination of a continuous function* of r and z, $v(r, z)$ such that

$$\frac{\partial v}{\partial z} = a\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) \tag{154}$$

in the region

$$\begin{pmatrix} z > 0 \\ 0 < r < 1 \end{pmatrix}, \tag{155}$$

and

$$l\frac{\partial v}{\partial r} + hv = 0 \qquad (z > 0, r = 1), \tag{156}$$

$$v = f(r) \qquad (z = 0), \tag{157}$$

where a, l, and h are constants and $f(r)$ is a function of r that gives the initial distribution of the temperature.†

It can be shown easily that the function

$$Ae^{-a\lambda^2 z}F_0(\lambda, r) \tag{158}$$

where A is a constant and λ is a root of the equation

$$l\lambda J_0'(\lambda) + hJ_0(\lambda) = 0 \tag{159}$$

satisfies conditions (154) and (156). It remains to construct from functions of the form (158) a continuous function which will also satisfy (154) and (156) and in addition will approach continuously the boundary value $f(r)$, where $f(r)$ is continuous, and which will remain finite where that is not the case.

Consider the series

$$A_1 e^{-a\lambda_1^2 z}F_0(\lambda_1, r) + A_2 e^{-a\lambda_2^2 z}F_0(\lambda_2, r) + \cdots \tag{160}$$

where the λ's are the roots, positive or zero, of equation (107) for $\nu = 0$ arranged in order of increasing magnitude and the A's are the coefficients of the series of the form (106) corresponding to the given function $f(r)$. If the function $f(r)$ is finite and integrable in the interval $0 \leqq r \leqq 1$, we obtain from (46), (50), and (105a)

$$|A_n F_0(\lambda_n, r)| < K\sqrt{\lambda_n} \qquad (0 \leqq r \leqq 1),$$

where K is a positive constant. Hence for the general term of (160) we have in view of the first inequality in (23)

* By a continuous function we mean that $v(r, z)$ is to be continuous in the two variables r and z throughout the region

$$\begin{pmatrix} z > 0 \\ 0 \leqq r \leqq 1 \end{pmatrix}$$

and is to approach the boundary value $f(r)$ continuously wherever $f(r)$ is continuous.

† Cf. CARSLAW, *Introduction to the Theory of Fourier's Series and Integrals, etc.*, Chapter XV.

$$|A_n e^{-a\lambda_n^2 z} F_0(\lambda_n, r)| < \frac{K\sqrt{\lambda_n}}{a\lambda_n^2 z} = \frac{K}{a\lambda_n^{\frac{3}{2}} z} < \frac{K}{aC^{\frac{3}{2}} n^{\frac{3}{2}} z} \leqq \frac{K}{aC^{\frac{3}{2}} z_0 n^{\frac{3}{2}}} * \quad \begin{pmatrix} z \geqq z_0 \\ 0 \leqq r \leqq 1 \end{pmatrix}$$

and therefore the series (160) is uniformly convergent in the region

$$\begin{pmatrix} 0 \leqq r \leqq 1 \\ z \geqq z_0 > 0 \end{pmatrix}. \tag{161}$$

Since, moreover, the terms of (160) are continuous functions of r and z for all values of r and z, it follows that the series defines a continuous function of r and z throughout the region (161), and therefore, since z_0 is any constant greater than zero, throughout the region

$$\begin{pmatrix} 0 \leqq r \leqq 1 \\ z > 0 \end{pmatrix}. \tag{162}$$

In similar fashion it may be shown that the function defined by (160) satisfies conditions (154) and (156). It only remains to consider the behavior of the function as we approach the line $z = 0$ from the positive side.

We know from Theorem V that if $f(r)$ is a function that is finite and integrable in the interval $0 \leqq r \leqq 1$, then the series

$$A_1 F_0(\lambda_1, r) + A_2 F_0(\lambda_2, r) + \cdots, \tag{163}$$

where the A's and λ's are those of the series (160), will be uniformly summable to the value of $f(r)$ throughout every interval that lies within a larger interval in which $f(r)$ is continuous. It follows then from Theorem VII† that, as z approaches zero through positive values, the function defined by (160) will approach the value $f(r)$ uniformly throughout any r-interval lying in a larger interval in which $f(r)$ is continuous. Consequently, if r' is any interior point of an interval in which $f(r)$ is continuous, the function defined by (160) approaches the value $f(r')$ as we approach the point $(r = r', z = 0)$ along any path lying in the region (155).

We now wish to examine the behavior of the function defined by (160) when we approach a point that is not an interior point of an interval of continuity of $f(r)$. If we define $S_n(r)$ for the series (163) in the same way that $S_n(x)$ was defined in (34) for a corresponding series, we may, for values of $z > 0$, write the series (160) in the form‡

$$\sum_{n=1}^{\infty} n S_n(r)[e^{-a\lambda_n^2 z} - 2e^{-a\lambda_{n+1}^2 z} + e^{-a\lambda_{n+2}^2 z}]. \tag{164}$$

* If $\lambda_1 = 0$ we must take the $(n+1)$st term as the general term instead of the nth term in order that this inequality may hold.

† In this theorem we did not consider the possibility of λ_1 being equal to zero. However, this is an unessential point, as the behavior of a finite number of convergence factors at the beginning of the series does not matter, provided they approach 1 as the parameter of which they are functions approaches zero.

‡ Cf. equation (7) of the writer's article in these Transactions referred to on page 430.

If now we make the change of variable

$$az = \alpha^2,$$

the expression (164) reduces to

$$\sum_{n=1}^{\infty} nS_n(r)[e^{-\lambda_n^2 a^2} - 2e^{-\lambda_{n+1}^2 a^2} + e^{-\lambda_{n+2}^2 a^2}]. \tag{165}$$

But we have from the corollary of Theorem II*

$$|S_n(r)| < M \quad (n = 1, 2, 3, \cdots; 0 < r_0 \leqq r < 1),$$

where M is a positive constant. Consequently, since the convergence factors (151) satisfy the conditions of BROMWICH's theorem, the series (165) is, throughout the region

$$\begin{pmatrix} \alpha > 0 \\ 0 < r_0 \leqq r \leqq 1 \end{pmatrix},$$

less in absolute value than

$$M \sum_{n=1}^{\infty} n \,|\, e^{-\lambda_n^2 a^2} - 2e^{-\lambda_{n+1}^2 a^2} + e^{-\lambda_{n+2}^2 a^2} \,| < MK,$$

where K is a positive constant. Hence if we approach any point $(r = r', t = 0)$ such that

$$0 < r_0 < c \leqq r' \leqq 1,$$

along any path lying in the region (155), the function defined by (160) remains finite.

Problem II. The problem of determining the steady flow of heat in a semi-infinite circular cylinder whose base is kept at a temperature

$$v = f(r),$$

where r is the perpendicular distance from the axis of the cylinder, reduces to the determination of a continuous function† of r and z which satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{\partial^2 v}{\partial z^2} = 0 \tag{166}$$

in the region (155), and also satisfies (156) and (157).‡

It can be shown easily that the function

$$Ae^{-\lambda z}F_0(\lambda, r),$$

where A is a constant and λ is a root of equation (159), satisfies (166) and (156). We form now the series

* This corollary was proved with regard to the series (35), but as (35) differs from (106) by at most one term that is finite for all values of the variable involved, it holds for the latter series also.

† Cf. note, p. 432.

‡ Cf. CARSLAW, loc. cit., p. 321

$$A_1 e^{-\lambda_1 z} F_0(\lambda_1, r) + A_2 e^{-\lambda_2 z} F_0(\lambda_2, r) + \cdots, \tag{167}$$

where the λ's are the roots, positive or zero, of equation (107) for $\nu = 0$ arranged in increasing order of magnitude, and the A's are the coefficients of the series of the form (106) corresponding to the given function $f(r)$.

If $f(r)$ is finite and integrable in the interval $0 \leqq r \leqq 1$, the series (167) will converge for $z > 0$ to a continuous function of r and z that satisfies (166) and (156), and approaches the proper boundary value when we approach a point on the line $z = 0$ that lies within an interval of continuity of $f(r)$. Furthermore the series (167) remains finite when we approach any point on the line $z = 0$ that lies in the interval

$$0 < c \leqq r \leqq 1.$$

The proof of the above facts is analogous to the proof in the previous problem, except that we use Theorem VI instead of Theorem VII.

THE UNIVERSITY OF CINCINNATI,
CINCINNATI, OHIO.

SINGULAR POINTS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS*

BY

GEORGE D. BIRKHOFF

Let

$$\frac{dy_i}{dx} = \sum_{j=1}^{n} a_{ij}(x) y_j \qquad (i=1, 2, \cdots, n) \tag{1}$$

be a system of n ordinary linear differential equations of the first order in which we have for $|x| > R$

$$a_{ij}(x) = a_{ij} x^q + a_{ij}^{(1)} x^{q-1} + \cdots + a_{ij}^{(q)} + a_{ij}^{(q+1)} \frac{1}{x} + \cdots \qquad (i, j = 1, 2, \cdots, n), \tag{2}$$

q being an integer. We propose to investigate the nature of the solutions of (1) in the vicinity of $x = \infty$, which is taken to be a singular point of the given system of equations so that $q \geqq -1$. The integer $q+1$ is termed the *rank* of (1) at $x = \infty$, after POINCARÉ.† It will be further assumed that the n roots

$$\alpha_1, \alpha_2, \cdots, \alpha_n$$

of the *characteristic equation*

$$|a_{ij} - \delta_{ij}\alpha| = 0 \tag{3}$$

[where $\delta_{ij} = 0$ $(i \neq j)$, and $\delta_{ii} = 1$] are distinct, and in the case $q = -1$ do not differ by integers. The writer hopes to discuss the general case later.

It is apparent that any proper linear transformation

$$\bar{y}_i = \sum_{j=1}^{n} \lambda_{ij}(x) y_j, \tag{4}$$

in which the functions $\lambda_{ij}(x)$ are *analytic or have a pole at* $x = \infty$, will transform (1) into a system of equations $(\bar{1})$ with coefficients $\bar{a}_{ij}(x)$ of the same form (2), although $\bar{q}$ is not necessarily equal to q. Any such system of equations $(\bar{1})$ will be said to be *equivalent to* (1) *at* $x = \infty$.

The first part of the paper deals with the determination of the simplest system of differential equations which is equivalent to (1) at $x = \infty$. In § 1 an important *Lemma on Analytic Functions* is stated and proved. The applica-

* Presented to the Society (Chicago), April 17, 1908, under a different title.

† Acta Mathematica, vol. 8 (1886), p. 305.

tion of this lemma in § 2 shows that there always exists a *canonical* system of equations equivalent to (1) at $x = \infty$ having the form

$$x\frac{dy_i}{dx} = \sum_{j=1}^{n} P_{ij}(x)y_j \qquad (i=1, 2, \cdots, n),$$

in which $P_{ij}(x)$ are polynomials. In § 3 it is shown that the rank of this canonical system of equations at $x = \infty$ can be taken to be q so that the degree of the polynomials $P_{ij}(x)$ does not exceed $q+1$.

The significance of these results is best seen by considering the case of a regular singular point $q = -1$. The canonical system of equations equivalent to (1) is then in simplest form (§ 3)

$$x\frac{dY_1}{dx} = \alpha_1 Y_1, \qquad x\frac{dY_2}{dx} = \alpha_2 Y_2, \ \cdots, \ x\frac{dY_n}{dx} = \alpha_n Y_n;$$

and a fundamental system of solutions is

$$\begin{aligned}
&Y_1^{(1)} = x^{\alpha_1}, \quad Y_2^{(1)} = 0, \quad \cdots, \quad Y_n^{(1)} = 0,\\
&Y_1^{(2)} = 0, \quad Y_2^{(2)} = x^{\alpha_2}, \quad \cdots, \quad Y_n^{(2)} = 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&Y_1^{(n)} = 0, \quad Y_2^{(n)} = 0, \quad \cdots, \quad Y_n^{(n)} = x^{\alpha_n}.
\end{aligned}$$

Now the solutions of (1) and the above system of equations are connected by a relation

$$y_i = \sum_{j=1}^{n} \lambda_{ij}(x)Y_j \qquad (i=1, 2, \cdots, n),$$

in which the $\lambda_{ij}(x)$ are analytic at $x = \infty$ (or have a pole, but this case is easily excluded). It follows that there exists a fundamental system of solutions of (1), $y_1^{(j)}, y_2^{(j)}, \cdots, y_n^{(j)}$ $(j = 1, 2, \cdots, n)$, of the form

$$y_i^{(j)} = \sum_{k=1}^{n} \lambda_{ik}(x)Y_k^{(j)} = \lambda_{ij}(x)x^{\alpha_j} \qquad (i, j=1, 2, \cdots, n).$$

This is the fundamental theorem for a regular singular point. *The solutions of the canonical system of equations equivalent to* (1) *at* $x = \infty$ *play the same rôle in the case of an irregular singular point* ($q \geqq 0$) *that they do in the case above considered.*

The nature of the solutions of the canonical system of equations can be discussed by special and elegant methods when $q \geqq 0$. In the case $q = 0$ it is known that the solutions of any system (1) with rational coefficients $a_{ij}(x)$ can be expressed by a Laplace integral

$$y_i = \int e^{xz}\eta_i(z)\,dz \qquad (i=1, 2, \cdots, n),$$

and that from this formula can be deduced the behavior of the solutions in the vicinity of $x = \infty$. This method of treatment is due to POINCARÉ,* and has been developed more fully by HORN.† It is applicable in an extremely simple form to canonical systems of equations for which $q = 0$. If $q > 0$, generalized Laplace integrals of a simple character will be shown to be available. The integrals for $q \geqq 0$ are considered in § 4.

These integrals are employed in § 5 to find the asymptotic nature of the solutions of the canonical system of equations equivalent to (1) at $x = \infty$ in certain sectors of the x-plane. The asymptotic form of the solutions of (1) in the same sectors follows as an immediate consequence. The powerful and general method of successive approximation which HORN ‡ has employed in various cases will no doubt lead to the same results.

A Riemannian characterization of the solutions of the canonical system of equations in the vicinity of $x = \infty$ is given and extended in § 6 to the solutions of (1). The number of characteristic constants in the solutions is found to be the same as the number of arbitrary constants of the canonical system of equations.

The characterization suggests the probable generalization of the Riemannian problem when the singular points of the system of differential equations are not regular. This formulation and a count of constants is given in § 7.

§ 1. *Auxiliary lemma.*

In order to obtain the simplest possible system $(\bar{1})$ equivalent to (1), we can choose as we please the n^2 functions λ_{ij} of the transformation (4) but with the restriction that they are analytic or have poles at $x = \infty$ and that the determinant $|\lambda_{ij}|$ is not identically zero. How can one choose the functions λ_{ij} so as to free as far as possible the solutions of $(\bar{1})$ from singularities in the finite plane? The answer to this question, which will be fundamental in our study of (1), is derived from the following lemma.

LEMMA ON ANALYTIC FUNCTIONS. *Let $\theta_{ij}(x)$ be a set of n^2 functions of x, single-valued and analytic for $|x| > R$ but not necessarily analytic at $x = \infty$, and also such that the determinant*

$$|\theta_{ij}(x)| \neq 0 \qquad (|x| > R).$$

Then it is possible to so choose n^2 multipliers $\lambda_{ij}(x)$, analytic at $x = \infty$, that

$$\sum_{j=1}^{n} \lambda_{ij}(x)\theta_{jk}(x) = x^{-m}\xi_{ik}(x), \qquad \left(\text{or } \sum_{j=1}^{n} \theta_{ij}(x)\lambda_{jk}(x) = x^{-m}\xi_{ik}(x)\right)$$

$$(i, k = 1, 2, \cdots, n),$$

*American Journal of Mathematics, vol. 7 (1885), pp. 203–258. A single linear differential equation of the nth order is treated in this and other papers referred to later.

†Mathematische Annalen, vol. 50 (1897), pp. 525–556.

‡Crelle's Journal, vol. 133 (1907), pp. 19–67.

where m is a fixed positive integer or zero, and the $\xi_{ik}(x)$ are entire functions of x for which the determinant $|\xi_{ik}(0)| \neq 0$.

Proof. The two alternative forms of the Lemma are clearly equivalent. We shall confine ourselves to the proof of the second of these, which is the easier to give in matrix notation.

In order to elucidate the lemma let us consider first the case $n = 1$. We have to determine λ_{11} so that

$$\theta_{11}\lambda_{11} = x^{-m}\xi_{11}.$$

Since $\theta_{11}(x) \neq 0$ for $|x| > R$, the function $\log \theta_{11}$ is expansible in a Laurent series for $|x| > R$ except for a term $\alpha \log x$,

$$\log \theta_{11} = \sum_{v=-\infty}^{+\infty} r_v x^v + \alpha \log x.$$

Take then

$$\lambda_{11} = e^{-\sum_{v=-1}^{-\infty} r_v x^v}, \qquad m = \alpha, \qquad \xi_{11} = e^{\sum_{v=0}^{+\infty} r_v x^v},$$

if $\alpha < 0$, or

$$\lambda_{11} = e^{-\sum_{v=-1}^{-\infty} r_v x^v}, \qquad m = 0, \qquad \xi_{11} = x^{\alpha} e^{\sum_{v=0}^{+\infty} r_v x^v},$$

if $\alpha \geqq 0$, and we have the desired solution. This explicit treatment is not possible if $n > 1$, in which case we shall have to construct a solution by an infinite process.

As it is convenient to use matrix notation, let Θ denote the matrix $\|\theta_{ij}\|$ and Θ^{-1} its inverse matrix $\|\theta_{ij}^{-1}\|$ which is of the same form, since $|\theta_{ij}| \neq 0$ for $|x| > R$. Also we shall write $\Lambda = \|\lambda_{ij}\|$.

Furthermore, if ϕ is single-valued and analytic in the vicinity of $x = \infty$, so that it is expansible in a Laurent series,

$$\phi = \sum_{v=-\infty}^{+\infty} \alpha_v x^v,$$

we write

$$P(\phi) = \sum_{v=0}^{+\infty} \alpha_v x^v, \qquad N(\phi) = \sum_{v=-1}^{-\infty} \alpha_v x^v, \qquad N_m(\phi) = \sum_{v=-m}^{-\infty} \alpha_v x^v.$$

Here m is some positive integer whose value will be specified later. We have the obvious relations

$$\phi = P(\phi) + N(\phi) = P(\phi) + \alpha_{-1}x^{-1} + \cdots + \alpha_{-m+1}x^{-m+1} + N_m(\phi),$$

$$P[N(\phi)] = P[N_m(\phi)] = N[P(\phi)] = N_m[P(\phi)] = 0.$$

More generally, if we have a matrix $\Phi \equiv \|\phi_{ij}\|$, we shall use the notation

$$P(\Phi) \equiv \|P(\phi_{ij})\|, \qquad N(\Phi) \equiv \|N(\phi_{ij})\|, \qquad N_m(\Phi) \equiv \|N_m(\phi_{ij})\|.$$

Let now $T = \| t_{ij} \|$ be any matrix of constants of determinant not zero. Consider the matrix equations

$$\Gamma = P(\Theta^{-1} \cdot \Psi) \qquad \Psi = Tx^{-m} + N_m(\Theta \cdot \Gamma). \tag{5}$$

If a solution of these equations $\Gamma = \| \gamma_{ij} \|$, $\Psi = \| \psi_{ij} \|$ *exists, we will have at once the desired matrix*

$$\Lambda = N(\Theta^{-1} \cdot \Psi). \tag{6}$$

To verify this statement we notice first that *the functions* λ_{ij} *are analytic at* $x = \infty$ by (6), since they are convergent series of negative powers of x. Moreover from the first equation of (5) we obtain

$$\Gamma + N(\Theta^{-1} \cdot \Psi) = \Theta^{-1} \cdot \Psi. \tag{7}$$

From this equation (7) we derive

$$\Theta \cdot \Gamma + \Theta[N(\Theta^{-1} \cdot \Psi)] = \Psi \tag{8}$$

on multiplying by the matrix Θ. Furthermore, by the second equation of (5), we obtain also

$$\Psi = Tx^{-m} + \Theta \cdot \Gamma - P(\Theta \cdot \Gamma) - R, \tag{9}$$

in which the elements of R are polynomial of degree $m - 1$ at most in $1/x$. If we add (8) and (9), we find by means of (6)

$$\Theta \cdot \Lambda = Tx^{-m} - P(\Theta \cdot \Gamma) - R.$$

The right hand member forms a matrix $\| x^{-m} \xi_{ij}(x) \|$ in which the $\xi_{ij}(x)$ are entire functions of x with the desired property

$$|\xi_{ij}(0)| = |t_{ij}| \neq 0.$$

We have thus proved our statement.

It remains to construct a solution of equations (5) *for some* m. This we do by a method of successive approximation.

Write

$$\begin{aligned} \Psi_0 &= Tx^{-m}, \\ \Gamma_0 &= P(\Theta^{-1} \cdot \Psi_0), \\ \Psi_1 &= Tx^{-m} + N_m(\Theta \cdot \Gamma_0), \\ \Gamma_1 &= P(\Theta^{-1} \cdot \Psi_1), \\ \Psi_2 &= Tx^{-m} + N_m(\Theta \cdot \Gamma_1), \\ &\cdots\cdots\cdots \end{aligned} \tag{10}$$

If the sequence of matrices Ψ_p, Γ_p tend to limit matrices, it will not be difficult to prove that these limits form the desired solutions.

Define the difference matrices

$$\overline{\Gamma}_p = \Gamma_p - \Gamma_{p-1}, \qquad \overline{\Gamma}_0 = \Gamma_0,$$
$$\overline{\Psi}_p = \Psi_p - \Psi_{p-1}, \qquad \overline{\Psi}_0 = \psi_0.$$

These matrices for $p > 0$ satisfy the equations

$$\overline{\Psi}_p = N_m(\Theta \cdot \overline{\Gamma}_{p-1}), \qquad \overline{\Gamma}_p = P(\Theta^{-1} \cdot \overline{\Psi}_p). \tag{11}$$

It follows from the first of these equations that $\overline{\Psi}_p$ contains only negative powers of x. Hence the positive powers which come from the multiplication of Θ^{-1} and $\overline{\Psi}_p$ in the second of these equations arise from the positive powers in the elements of Θ^{-1} and from the negative in $\overline{\Psi}_p$ so that

$$P(\Theta^{-1} \cdot \overline{\Psi}_p) = P[P(\Theta^{-1}) \cdot \overline{\Psi}_p]. \tag{12}$$

Likewise we find

$$N_m(\Theta \cdot \overline{\Gamma}_{p-1}) = N_m[N_m(\Theta) \cdot \overline{\Gamma}_{p-1}], \tag{13}$$

since the elements of $\overline{\Gamma}_{p-1}$ contain only positive powers of x and a constant term. Thus (11) can be written

$$\overline{\Psi}_p = N_m[N_m(\Theta) \cdot \overline{\Gamma}_{p-1}], \qquad \overline{\Gamma}_p = P[P(\Theta^{-1}) \cdot \overline{\Psi}_p]. \tag{14}$$

In this form we shall prove the convergence of

$$\Gamma = \overline{\Gamma}_0 + \overline{\Gamma}_1 + \cdots, \qquad \Psi = \overline{\Psi}_0 + \overline{\Psi}_1 + \cdots$$

for large enough m.

Choose r_0, R_0 so that

$$r_0 > R_0 > R;$$

then we can choose M so large that*

$$P(\theta_{ij}^{-1}) \ll \frac{M}{1 - \dfrac{x}{r_0}} = M\left(1 + \frac{x}{r_0} + \frac{x^2}{r_0^2} + \cdots\right) \tag{15}$$

since $P(\theta_{ij}^{-1})$ is an entire function of x. Moreover since $N(\theta_{ij})$ is analytic for $|x| > R$, the series for $N(\theta_{ij})$,

$$\frac{\theta_{ij}^{(1)}}{x} + \frac{\theta_{ij}^{(2)}}{x^2} + \cdots,$$

* The notation, $\alpha \ll \beta$, means that the coefficients of the various powers of x in the power series α do not exceed the corresponding terms of β in absolute value. I shall extend this notation of POINCARÉ to matrices A and B of which corresponding terms α_{ij} and β_{ij} are power series. Thus the notation $A \ll B$ implies that $\alpha_{ij} \ll \beta_{ij}$ for all values of i and j.

is convergent for $|x| > R$, so that

$$\lim_{p=\infty} \frac{\theta_{ij}^{(p)}}{R_0^p} = 0.$$

Hence for large enough m we have $|\theta_{ij}^{(p)}| < \epsilon R_0^p$, $p > m$, no matter how small ϵ be taken. Thus we can choose m so large that for any small ϵ

$$N_m(\theta_{ij}) \ll \frac{\epsilon}{\frac{x}{R_0} - 1} = \epsilon\left(\frac{R_0}{x} + \frac{R_0^2}{x^2} + \cdots\right). \tag{16}$$

Take m so large that

$$\frac{n^2 \epsilon M r_0 R_0}{(r_0 - R_0)^2} < 1. \tag{17}$$

It is to be noticed that both the comparison series used converge for $r_0 > |x| > R_0$.

Now choose L so large that

$$\overline{\Gamma}_0 = \Gamma_0 \ll \left|\left| \frac{L}{1 - \frac{x}{r_0}} \right|\right|. \tag{18}$$

First by (16) and (18) we see that

$$\overline{\Psi}_1 = N_m[N_m(\Theta)\cdot\Gamma_0]$$

$$\ll \left|\left| N_m\left(\frac{n\epsilon L r_0 R_0}{(x - R_0)(r_0 - x)}\right)\right|\right| = \left|\left| N_m\left[\frac{n\epsilon L r_0 R_0}{(r_0 - R_0)}\left(\frac{1}{x - R_0} + \frac{1}{r_0 - x}\right)\right]\right|\right|$$

$$\ll \left|\left| \frac{n\epsilon L r_0 R_0}{(r_0 - R_0)(x - R_0)} \right|\right|.*$$

Next, by (15) and the above inequality, we see that

$$\overline{\Gamma}_1 = P[P(\Theta^{-1})\cdot\overline{\Psi}_1] \ll \left|\left| P\left[\frac{n^2 \epsilon M L r_0^2 R_0}{(r_0 - R_0)(r_0 - x)(x - R_0)}\right]\right|\right| \ll \left|\left| \frac{n^2 \epsilon M L r_0^2 R_0}{(r_0 - R_0)^2 (r_0 - x)} \right|\right|$$

as before. And thus in general

$$\overline{\Psi}_p \ll \left|\left| \frac{n^{2p-1} \epsilon^p M^{p-1} L r_0^p R_0^p}{(r_0 - R_0)^{2p-1}(x - R_0)} \right|\right|, \qquad \overline{\Gamma}_p \ll \left|\left| \frac{n^{2p} \epsilon^p M^p L r_0^{p+1} R_0^p}{(r_0 - R_0)^{2p}(r_0 - x)} \right|\right|,$$

as may be proved by induction. If (17) holds, we shall have *convergent series* for γ_{ij} and ψ_{ij} if $r_0 > |x| > R_0$.

Furthermore, from the equations (10) we infer that

$$\Gamma = P(\Theta^{-1}\cdot\Psi),$$

* The compound matrix has n terms in each element; hence we get a multiplier n.

so that γ_{ij} must be an entire function of x. Then likewise we infer that

$$\Psi = \Gamma x^{-m} + N_m(\Theta \cdot \Gamma).$$

Thus $||\gamma_{ij}||$ and $||\psi_{ij}||$ satisfy (5).

§ 2. *Existence of an equivalent canonical system.*

We are now in a position to answer the question which was raised at the beginning of § 1.

By the hypothesis concerning (2) there are no finite singularities of the functions $a_{ij}(x)$ of (1) exterior to the circle $|x| = R$. Let x describe a closed path exterior to this circle about $x = \infty$. It is easy to show that there will exist n linearly independent particular solutions

$$\begin{array}{cccc} y_1^{(1)}, & y_2^{(1)}, & \cdots, & y_n^{(1)}, \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n)}, & y_2^{(n)}, & \cdots, & y_n^{(n)}, \end{array}$$

which undergo the substitution

$$(19) \qquad \begin{array}{ccccc} \bar{y}_1^{(1)} = \rho_1 y_1^{(1)}, & \bar{y}_2^{(1)} = \rho_1 y_2^{(1)}, & \cdots, & \bar{y}_n^{(1)} = \rho_1 y_n^{(1)}, \\ \cdot & \cdot & \cdot & \cdot \\ \bar{y}_1^{(n)} = \rho_n y_1^{(n)}, & y_2^{(n)} = \rho_n y_2^{(n)}, & \cdots, & \bar{y}_n^{(n)} = \rho_n y_n^{(n)}, \end{array}$$

unless two or more values of ρ are equal, when degenerate cases may occur.*

We will suppose for the present that there exists a fundamental system of this character (19) for which, of course,

$$|y_i^{(j)}| = c\, e^{\int [a_{11}(x) + a_{22}(x) + \cdots + a_{nn}(x)]dx}.$$

Define now functions $\theta_{ij}(x)$ by the n^2 equations

$$y_i^{(j)} = x^{k_j}\theta_{ij}(x), \qquad \left(k_j = \frac{1}{2\pi\sqrt{-1}} \log \rho_j\right).$$

It may be inferred at once that the θ_{ij} are single valued and analytic for $|x| > R$ and that

$$|\theta_{ij}(x)| = c\, x^{-(k_1 + \cdots + k_n)}\, e^{\int [a_{11}(x) + a_{22}(x) + \cdots + a_{nn}(x)]dx}.$$

Thus the $\theta_{ij}(x)$ fulfill the restrictions of the Lemma. Hence we can determine a set of functions $\lambda_{ij}(x)$ analytic at $x = \infty$ in such a manner that

$$\sum_{j=1}^{n} \lambda_{ij}(x)\theta_{jk}(x) = x^{-m}\xi_{ik}(x),$$

the $\xi_{ik}(x)$ being entire functions of x and $|\xi_{ik}(0)| \neq 0$.

* It is here simply a matter of the canonical form of a linear substitution.

The transformation (4) changes (1) into a new system of equations ($\bar{1}$) with a fundamental system of solutions

$$\bar{y}_i^{(j)} = x^{k_j - m}\xi_{ij}(x) \qquad (i, j = 1, 2, \cdots, n).$$

Therefore we can transform a system (1) *into an equivalent system whose solutions have no other singularity in the finite x-plane except* $x = 0$, *and this singularity will be of the above simple form.*

The new system of equations can be written

$$(20) \qquad \frac{1}{|\bar{y}_i^{(j)}|} \cdot \begin{vmatrix} \frac{d\bar{y}_i}{dx} & \bar{y}_1 & \bar{y}_2 & \cdots & \bar{y}_n \\ \frac{d\bar{y}_i^{(1)}}{dx} & \bar{y}_1^{(1)} & \bar{y}_2^{(1)} & \cdots & \bar{y}_n^{(1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{d\bar{y}_i^{(n)}}{dx} & \bar{y}_1^{(n)} & \bar{y}_2^{(n)} & \cdots & \bar{y}_n^{(n)} \end{vmatrix} = 0 \quad (i = 1, 2, \cdots, n).$$

Hence we see that the coefficients $\bar{a}_{ij}(x)$ are quotients of entire functions, and therefore meromorphic in the finite plane with poles only at points for which

$$\bar{\Delta} = |\bar{y}_i^{(j)}| = 0.$$

On the other hand, by a fundamental property of any transformation (4), the new system of equations (20) is of the same form as (1) so that the $\bar{a}_{ij}(x)$ are analytic at $x = \infty$, or have a pole at that point.*

These two properties of the functions $\bar{a}_{ij}$ show that they are rational functions of x. *Thus the new system of equations equivalent to* (1) *at* $x = \infty$ *can be written*

$$(21) \qquad \frac{dy_i}{dx} = \sum_{j=1}^{n} R_{ij}(x) y_j,$$

in which the coefficients $R_{ij}(x)$ *are rational.* Furthermore it possesses a fundamental system of solutions

$$(22) \qquad y_i^{(j)} = x^{k_j - m}\xi_{ij}(x),$$

where the $\xi_{ij}(x)$ are entire functions of x and $|\xi_{ij}(0)| \neq 0$, and the poles of $R_{ij}(x)$ lie at points for which

$$\Delta = |y_i^{(j)}| = 0.$$

* Write the transformation inverse to (4) in the form

$$y_i = \sum_{j=1}^{n} \bar{\lambda}_{ij}\, \bar{y}_j,$$

in which, of course, the $\bar{\lambda}_{ij}(x)$ are analytic at $x = \infty$, or have a pole at that point. A direct computation shows that we have

$$\bar{a}_{ij} = \sum_{k,l=1}^{n} \lambda_{ik} a_{kl} \bar{\lambda}_{lj} + \sum_{k=1}^{n} \bar{\lambda}_{kj} \frac{d}{dx} \lambda_{ik}.$$

By a succession of transformations (4) we shall now remove all the singular points of (21), not at $x = 0$ and $x = \infty$. These can only be *apparently* singular points because of the character of the fundamental system of solutions (22).

Since from (21) we have

$$\Delta = ce^{\int [R_{11}(x)+R_{22}(x)+\cdots+R_{nn}(x)]dx}$$

it is clear that the entire function Δ is the product of a factor which never vanishes, and of factors of the form $(x - x')^\lambda$, λ being a positive integer. The singular points will lie at $x = x'$. These last factors can be removed as follows. Choose constants $c_1, c_2, \cdots, c_n$, not all zero, so that for $x = x'$

$$c_1 y_1^{(j)} + c_2 y_2^{(j)} + \cdots + c_n y_n^{(j)} = 0 \qquad (j = 1, 2, \cdots, n) \tag{23}$$

This is possible because by hypothesis $\Delta = 0$ for $x = x'$. Let c_k be one of the quantities c_i which is not zero. Make the further transformation of the form (4)

$$\bar{y}_i = y_i \qquad (i \neq k),$$

$$\bar{y}_k = \frac{c_1 y_1 + c_2 y_2 + \cdots + c_n y_n}{x - x'}.$$

The fundamental system of solutions $y_i^{(j)}$ of (21) is transformed into another such system of the same character,

$$\bar{y}_i^{(j)} = y_i^{(j)} \qquad (i \neq k),$$

$$\bar{y}_k^{(j)} = \frac{c_1 y_1^{(j)} + c_2 y_2^{(j)} + \cdots + c_n y_n^{(j)}}{x - x'}.$$

Accordingly the new system of equations has again the form (21). But we have

$$\bar{\Delta} = c_k \frac{\Delta}{x - x'},$$

so that a factor $x - x'$ has been removed from Δ.

In this way one can remove in succession each factor $x - x'$, and finally reach the case when Δ does not vanish except perhaps for $x = 0$. Thus a system of equations (21) exists whose solutions have the *additional property*

$$|y_i^{(j)}| \neq 0 \qquad (x \neq 0). \tag{24}$$

It follows that the coefficients $R_{ij}(x)$ cannot have any poles except at $x = 0$ and $x = \infty$, and so are either polynomials or polynomials multiplied by a positive integral power of $1/x$. The index of this power does not exceed 1. For suppose we have

$$R_{ij}(x) = \left(\frac{1}{x}\right)^r \{ t_{ij} + t_{ij}^{(1)} x + \cdots \} \qquad (i, j = 1, 2, \cdots, n),$$

in which $\tau > 1$ and not every t_{ij} is zero. Substituting $y_i^{(j)} = x^{k_j - n}\xi_{ij}(x)$ in (21) and equating coefficients of lowest powers of x, we obtain

$$0 = \sum_{k=1}^{n} t_{ij}\xi_{kj}(0) \qquad (i, k = 1, 2, \cdots, n).$$

By taking $i = 1, 2, \cdots, n$ successively it is inferred that $t_{ij} = 0$ $(i, j = 1, 2, \cdots, n)$ since $|\xi_{ij}(0)| \neq 0$. This is contrary to hypothesis. Therefore we must have

$$R_{ij}(x) = \frac{P_{ij}(x)}{x},$$

where $P_{ij}(x)$ denotes some polynomial in x.

THEOREM. *There always exists a canonical system of equations equivalent to* (1) *at* $x = \infty$,

$$x\frac{dy_i}{dx} = \sum P_{ij}(x)y_j \qquad (i = 1, 2, \cdots, n), \tag{25}$$

in which the $P_{ij}(x)$ *are polynomials.*

The proof of this theorem has been based on the assumption that there exist n linearly independent solutions of (1) of the character (19). The extension of the above proof to the degenerate cases is illustrated by taking a degenerate case for which $\rho_{n-1} = \rho_n$. Then there exist n linearly independent solutions such that we have in place of (19)

$$\begin{aligned}
&\bar{y}_1^{(1)} = \rho_1 y_1^{(1)}, && \bar{y}_2^{(1)} = \rho_1 y_2^{(1)}, && \cdots, && \bar{y}_n^{(1)} = \rho_1 y_n^{(1)},\\
&\cdots\cdots\cdots

made, the new system of equations in $\bar{y}_i$ will have a fundamental system of solutions

$$\bar{y}_i^{(j)} = x^{k_j - m}\xi_{ij}(x) \qquad (i=1, 2, \cdots, n; j=1, 2, \cdots, n-1),$$

$$\bar{y}_i^{(n)} = x^{k_{n-1}-m}\left[\xi_{in}(x) + \frac{1}{2\pi\rho_{n-1}\sqrt{-1}}\xi_{i,n-1}(x)\right] \qquad (i=1, 2, \cdots, n),$$

in which $\xi_{ij}(x)$ is an entire function and $|\xi_{ij}(0)| \neq 0$. The rest of the argument in the degenerate case is very like that employed before.

§ 3. *Formal solutions and the polynomials* $P_{ij}(x)$.

It can be shown easily by means of the formal solutions of (1)* that the polynomials $P_{ij}(x)$ cannot all be of degree less than $q+1$.

Let us first consider these formal solutions in the case $q \geqq 0$. If we substitute in (1) formal expressions

$$y_i = e^{P(x)}x^{\kappa}A_i(x) \qquad (i=1, 2, \cdots, n),$$

$$P(x) = \alpha\frac{x^{q+1}}{q+1} + \beta\frac{x^q}{q} + \cdots + \lambda x,$$

$$A_i(x) = A_i + A_i^{(1)}\frac{1}{x} + \cdots,$$

and attempt to determine $\alpha, \beta, \cdots, \lambda, \kappa, A_i, A_i^{(1)}, \cdots$ so that the equations resulting are formally satisfied, and not all the n quantities A_i are zero, we shall find that α is one of the n roots of the characteristic equation (3) and that we can then determine

$$A_i, \beta, A_i^{(1)}, \cdots, \kappa, A_i^{(q+2)},$$

in succession; thus there arise *precisely* n formal solutions of (1)

$$y_1 = S_1^{(j)}, \; y_2 = S_2^{(j)}, \; \cdots, \; y_n = S_n^{(j)} \qquad (j=1, 2, \cdots, n),$$

where

$$S_i^{(j)} = e^{P_j(x)}x^{\kappa_j}A_{ij}(x) \qquad (i, j=1, 2, \cdots, n),$$

$$P_j(x) = \alpha_j\frac{x^{q+1}}{q+1} + \beta_j\frac{x^q}{q} + \cdots + \lambda_j x, \tag{26}$$

$$A_{ij}(x) = A_{ij} + A_{ij}^{(1)}\frac{1}{x} + \cdots.$$

The first set of equations determining A_{ij} are

$$A_{ij}\alpha_j = \sum_{k=1}^{n} a_{ik}A_{kj} \qquad (i, j=1, 2, \cdots, n). \tag{27}$$

*SCHLESINGER, *Vorlesungen über lineare Differentialgleichungen*, pp. 176–182. For a discussion of regular singular points see pp. 122–171.

It is necessary for us to know also that the formal determinant

$$|S_i^{(j)}| = e^{P_1(x)+\cdots+P_n(x)} x^{\kappa_1+\cdots+\kappa_n}\left\{D + D^{(1)}\frac{1}{x} + \cdots\right\}$$

does not vanish identically. As a matter of fact, the constant $D = |A_{ij}|$ is not zero. Otherwise we could find multipliers $m_1, m_2, \cdots, m_n$, not all zero, such that

$$\sum_{j=1}^{n} A_{ij} m_j = 0 \qquad (i = 1, 2, \cdots, n).$$

Multiplying equations (27) by $m_j (j = 1, 2, \cdots, n)$ and adding, we obtain

$$\sum_{j=1}^{n} A_{ij} \alpha_j m_j = 0 \qquad (i = 1, 2, \cdots, n).$$

A second set of multipliers is $m_j' = \alpha_j m_j$; likewise a third set is $m_j'' = \alpha_j^2 m_j$. Thus we obtain n sets of multipliers $m_j, \alpha_j m_j, \cdots, \alpha_j^{n-1} m_j$. It is clear then that we have

$$\sum_{j=1}^{n} A_{ij} m_j [c_0 + c_1 \alpha_j + \cdots + c_{n-1} \alpha_j^{n-1}] = 0$$

for all values of $c_0, c_1, \cdots, c_{n-1}$. Since $\alpha_1, \alpha_2, \cdots, \alpha_n$ are distinct we conclude that

$$A_{ij} m_j = 0.$$

Hence for any value of j for which $m_j \neq 0$ we have

$$A_{ij} = 0 \qquad (i = 1, 2, \cdots, n),$$

which is impossible.

Let us turn now to consider the formal solutions of any system $(\bar{1})$, equivalent to (1) at $x = \infty$ by means of a transformation (4). The formal solutions of $(\bar{1})$ are defined by the equations

$$\bar{S}_i^{(j)} = \sum_{k=1}^{n} \lambda_{ik} S_k^{(j)} \qquad (i, j = 1, 2, \cdots, n).$$

Hence $\bar{S}_i^{(j)}$ contains the same exponential factor as $S_i^{(j)}$, although $\bar{\kappa}_j$ may differ from κ_j by an integer. The formal determinant $|\bar{S}_i^{(j)}|$ is also not identically zero, since we have

$$|\bar{S}_i^{(j)}| = |\lambda_{ij}(x)| \cdot |S_i^{(j)}|.$$

The same results are obtained for the case $q = -1$ except that the exponential multipliers $e^{P_j(x)}$ are lacking in the formal expressions $S_i^{(j)}$ and $\bar{S}_i^{(j)}$.

We will now prove for the case $q \geqq 0$ that the rank of any system of equations $(\bar{1})$ equivalent to (1) at $x = \infty$ is *at least* $q + 1$. Otherwise we must

have equations like (27) ($a_{ij}^{(l)}$ being replaced by $\bar{a}_{ij}^{(l)}$) to determine the constants $\bar{A}_{ij}$ of the formal solutions of $(\bar{1})$, in which however

$$\bar{a}_{ij} = 0 \qquad (i, j = 1, 2, \cdots, n).$$

These equations are

$$\bar{A}_{ij}\alpha_j = 0 \qquad (i, j = 1, 2, \cdots, n),$$

whence we see that $\bar{A}_{ij} = 0$ $(i = 1, 2, \cdots, n)$ for any α_j not zero. This is impossible.

In particular, then, the rank of the canonical system (25) equivalent to (1) at $x = \infty$ is at least $q + 1$. *The polynomials $P_{ij}(x)$ cannot all be of degree less than $q + 1$.*

We shall now prove *that the degree of the polynomials $P_{ij}(x)$ can be taken not to exceed $q + 1$.*

Let us denote the n formal solutions of (25) by $T_1^{(j)}$, $T_2^{(j)}$, $\cdots$, $T_n^{(j)}$ $(j = 1, 2, \cdots, n)$ where

$$T_i^{(j)} = e^{P_j(x)} x^{l_j} B_{ij}(x),$$

$$B_{ij}(x) = B_{ij} + B_{ij}^{(1)}\frac{1}{x} + \cdots, \tag{28}$$

$\kappa_j - l_j$ being an integer. Let t_i be the index of the highest power of $1/x$ which all the $B_{ij}(x)$ $(j = 1, 2, \cdots, n)$ contain as a factor for fixed i. Then we may write

$$T_i^{(j)} = e^{P_j(x)} x^{l_j + t_i} C_{ij}(x),$$

$$C_{ij}(x) = C_{ij} + C_{ij}^{(1)}\frac{1}{x} + \cdots.$$

By a succession of transformations (4) of such a character as not to change the form of (25), we now proceed, if necessary, so to transform (25) that

$$|C_{ij}| \neq 0. \tag{29}$$

Assume the indices 1 to n so chosen that

$$t_1 \leqq t_2 \leqq \cdots \leqq t_n.$$

If $|C_{ij}| = 0$, one can choose constants $c_1, c_2, \cdots, c_n$ not all zero, such that

$$c_1 C_{1j} + c_2 C_{2j} + \cdots c_n C_{nj} = 0 \qquad (j = 1, 2, \cdots, n).$$

Let c_s be the last one of this set which is not zero. The transformation of the form (4),

$$\bar{y}_i = y_i \qquad (i \neq s),$$

$$\bar{y}_s = c_1 x^{t_s - t_1} y_1 + c_2 x^{t_s - t_2} y_2 + \cdots + c_s y_s,$$

leaves (25) unaltered in form. The new formal expressions $\bar{T}_i^{(j)}$ are the same as $T_i^{(j)}$ for $i \neq s$. Also

$$\bar{T}_s^{(j)} = c_1 x^{t_s - t_j} T_1^{(j)} + c_2 x^{t_s - t_i} T_2^{(j)} + \cdots + c_s T_s^{(j)}$$

$$= e^{P_j(x)} x^{t_j + t_s} \left\{ \sum_{l=0}^{s} c_l C_{ij} + \sum_{l=0}^{s} c_l C_{ij}^{(1)} \frac{1}{x} + \cdots \right\} \quad (j = 1, 2, \cdots, n).$$

The terms $\sum_{l=0}^{s} c_l C_{ij}$ $(j = 1, 2, \cdots, n)$ are all zero by hypothesis. The formal expressions $\bar{T}_s^{(j)}$ $(j = 1, 2, \cdots, n)$ will contain another factor $1/x$, and we have increased t_s by unity or more.

But we have

$$|\bar{T}_i^{(j)}| = c_s |T_i^{(j)}|$$

directly from the transformation. In this way we can increase $t_1, t_2, \cdots, t_n$ without altering, save by a constant factor, the formal determinant $|T_i^{(j)}|$. Since the determinant contains $1/x$ at least to the power

$$l_1 + l_2 + \cdots + l_n + t_1 + t_2 + \cdots + t_n,$$

we see that this process of transformation must finally come to an end. This occurs when (29) is satisfied.

When (25) has been transformed so that also (29) holds we may prove at once *that the degree* d_{ij} of P_{ij} *can not exceed* $q + 1 + t_i - t_j$. In fact, if some of the integers

$$d_{ij} - (q + 1 + t_i - t_j) \qquad (i, j = 1, 2, \cdots, n)$$

are positive, let τ be the greatest of them. Also denote the coefficients of the term

$$x^{\tau + q + 1 + t_i - t_j}$$

in $P_{ij}(x)$ by p_{ij}. Substitute the formal solutions in (25). When coefficients of highest powers of x are equated to zero one obtains

$$0 = \sum_{j=1}^{n} p_{ij} C_{jk} \qquad (i, k = 1, 2, \cdots, n),$$

which is not possible in view of (29).

We note at this point that the degree of $P_{ij}(x)$ does not exceed $q + 1$ if $i \leqq j$. We will now make further transformations so that this will also be true when $i > j$.

Let us first attempt to remove the constant terms of $P_{nj}(x)$ $(j = 1, 2, \cdots, n-1)$ by a transformation

$$\bar{y}_i = y_i \quad (i \neq n), \qquad \bar{y}_n = c_{nn} y_n + c_{n, n-1} y_{n-1} + \cdots + c_{n1} y_1,$$

where $c_{nn} = 1$. We will then obtain

$$\begin{aligned}
\overline{P}_{ij}(x) &= P_{ij}(x) - c_{nj} P_{in}(x) && (i, j \neq n),\\
\overline{P}_{in}(x) &= P_{in}(x) && (i \neq n),\\
\overline{P}_{nk}(x) &= \sum_{j=1}^{n} c_{nj} P_{jk}(x) - \sum_{j=1}^{n} c_{nj} P_{jn}(x) c_{nk} && (k \neq n),\\
\overline{P}_{nn}(x) &= \sum_{j=1}^{n} c_{nj} P_{jn}(x).
\end{aligned} \tag{30}$$

It is to be noted that the degree of $\overline{P}_{ij}(x)\,(i, j = 1, 2, \cdots, n)$ does not exceed $q + 1 + t_i - t_j$. If $p_{ij}^{(0)}$ denotes the constant term of $P_{ij}(x)$ we have to satisfy the conditions

$$\sum_{j=1}^{n} c_{nj} p_{jk}^{(0)} - \sum_{j=1}^{n} c_{nj} p_{jn}^{(0)} c_{nk} = 0 \qquad (k = 1, 2, \cdots, n-1).$$

Or if we write

$$\sum_{j=1}^{n} c_{nj} p_{jn}^{(0)} = \lambda = \lambda c_{nn},$$

these conditions may be stated in the form

$$\sum_{j=1}^{n} c_{nj} p_{jk}^{(0)} - \lambda c_{nk} = 0 \qquad (k = 1, 2, \cdots, n).$$

This gives the equations for λ,

$$|p_{ij}^{(0)} - \delta_{ij}\lambda| = 0.$$

Now each transformation employed since we obtained (21) has not affected the form (22) of the solutions. Hence (25) still possesses a fundamental system of solutions of this form. A direct substitution of these solutions in (25) will show that the roots λ are the quantities

$$k_1 - m, \cdots, k_n - m.$$

The roots λ are therefore distinct and do not differ by integers in the case in which (1) has n solutions of the form (19) where $\rho_1, \rho_2, \cdots, \rho_n$ are distinct, inasmuch as

$$k_j = \frac{1}{2\pi\sqrt{-1}} \log \rho_j \qquad (j = 1, 2, \cdots, n).$$

We confine ourselves to this simplest case.

Choose for λ any root $k_n - m$, determining a set of values $c_{nj}\,(j = 1, 2, \cdots, n)$ in which $c_{nn} \neq 0$. Such a root exists, since otherwise the linear equations

$$\sum_{j=1}^{n-1} c_{nj} p_{jk}^{(0)} - \lambda c_{nk} = 0 \qquad (k = 1, 2, \cdots, n-1)$$

could be satisfied for n values of λ, and there are only $n - 1$ such values. The corresponding transformation $(c_{nn} = 1)$ will remove the desired terms. Also we have $p_{nn}^{(0)} = k_n - m$ in the new system of equations.

Next we remove the constant terms of $P_{n-1,j}(x)$ $(j=1, 2, \cdots, n-2)$ by a transformation

$$\bar{y}_i = y_i \quad (i \neq n-1), \qquad \bar{y}_{n-1} = y_{n-1} + c_{n-1,n-2}y_{n-2} + \cdots + c_{n-1,1}y_1.$$

This can always be accomplished as before and we shall have reduced the constant term of $p_{n-1,n-1}(x)$ to $k_{n-1} - m$. Continuing thus, we finally *remove all the constant terms of* $P_{ij}(x)$ $(i<j)$ leaving

$$p_{11}^{(0)} = k_1 - m, \; p_{22}^{(0)} = k_2 - m, \; \cdots, \; p_{nn}^{(0)} = k_n - m.$$

When the constant terms are removed, we apply a transformation

$$\bar{y}_i = y_i \quad (i \neq \alpha), \qquad \bar{y}_\alpha = y_\alpha + cx^\tau y_\beta$$

to remove any term $p_{\alpha\beta}^{(\tau)}x^\tau$ of $P_{\alpha\beta}(x)$ $(\alpha>\beta)$ of degree τ less than $t_\alpha - t_\beta$. A direct computation shows that we have

$$\begin{aligned}
&\bar{P}_{ij}(x) = P_{ij}(x) && (i\neq\alpha, j\neq\beta),\\
&\bar{P}_{i\beta}(x) = P_{i\beta}(x) - cx^\tau P_{i\alpha}(x) && (i\neq\alpha),\\
&\bar{P}_{\alpha j}(x) = P_{\alpha j}(x) - cx^\tau P_{\beta j}(x) && (j\neq\beta),\\
&\bar{P}_{\alpha\beta}(x) = P_{\alpha\beta}(x) - cx^\tau [P_{\alpha\alpha}(x) - P_{\beta\beta}(x) + cx^\tau P_{\beta\alpha}(x)].
\end{aligned} \tag{31}$$

It is clear that the degree of $\bar{P}_{ij}(x)$ $(i, j=1, 2, \cdots, n)$ is not greater than $q+1+t_i-t_j$ since $\tau < t_\alpha - t_\beta$.

Let us take $\tau = 1$. All the new terms in $\bar{P}_{ij}(x)$ $(i<j)$ are at least of first degree, since a multiplier x appears throughout in them. Furthermore, since $P_{i\alpha}(x)$ $(i>\alpha)$ and $P_{\beta j}(x)$ $(\beta>j)$ contain no constant terms after the previous transformation, it is clear that there are no new terms of first degree except possibly in

$$\bar{P}_{i\beta}(x) \quad (i<\alpha), \qquad \bar{P}_{\alpha j}(x) \quad (\beta<j), \qquad \bar{P}_{\alpha\beta}(x).$$

Also the terms of the first degree in $\bar{P}_{\alpha\beta}(x)$ can be made zero by a proper choice of c, namely such that

$$p_{\alpha\beta}^{(1)} - c(k_\alpha - k_\beta) = 0.$$

This shows that we may remove the terms of first degree in $P_{ij}(x)$ $(t_i - t_j > 1)$, by taking them in proper order, namely with increasing i and decreasing j.

In the same way we may continue, taking $\tau = 2, 3, \cdots$, until all terms of $P_{ij}(x)$ $(i>j)$ of degree less than $t_i - t_j$ are removed. We may therefore write finally

$$P_{ij}(x) = x^{t_i - t_j}Q_{ij}(x)$$

where $Q_{ij}(x)$ is a polynomial for $i>j$. This holds likewise for $i \leqq j$, since then

$t_i - t_j \leqq 0$. Also $Q_{ij}(x)$ will be of degree $q+1$ or less, since $P_{ij}(x)$ is of degree $q+1+t_i-t_j$ or less.

The transformation

$$y_i = x^{t_i}\bar{y}_i \qquad (i=1, 2, \cdots, n)$$

will give us a system of equations

$$x\frac{d\bar{y}_i}{dx} = \sum_{j=1}^{n} \bar{P}_{ij}(x)\bar{y}_j,$$

in which

$$\bar{P}_{ij}(x) = x^{t_j-t_i}P_{ij} = Q_{ij}(x) \quad (i \neq j), \qquad \bar{P}_{ii}(x) = P_{ii}(x) + t_i = Q_{ii}(x) + t_i.$$

Thus there exists a canonical system (25), equivalent to (1) at $x=\infty$, in which the degree of $P_{ij}(x)$ does not exceed $q+1$.

We still have at our disposal a linear transformation with constant coefficients,

$$Y_i = \sum c_{ij}y_j.$$

In view of the nature of the roots of the characteristic equation (3) we may simplify $P_{ij}(x)$ by this means so that the equivalent canonical system of equations becomes*

$$x\frac{dY_i}{dx} = \sum_{j=1}^{n} P_{ij}(x)Y_j,$$

$$P_{ij}(x) = p_{ij}^{(0)} + p_{ij}^{(1)}x + \cdots + p_{ij}^{(q)}x^q \qquad (i \neq j), \tag{32}$$

$$P_{ii}(x) = p_{ii}^{(0)} + p_{ii}^{(1)}x + \cdots p_{ii}^{(q)}x^q + \alpha_i x^{q+1}.$$

The result which we have obtained is that *there exists a canonical system of equations of the form* (32) *equivalent to* (1) *at* $x=\infty$. This form will be hereafter assumed.

In the case $q=-1$ the canonical system reduces to

$$x\frac{dY_1}{dx} = \alpha_1 Y_1, \quad x\frac{dY_2}{dx} = \alpha_2 Y_2, \quad \cdots, \quad x\frac{dY_n}{dx} = \alpha_n Y_n.$$

But this system of equations is equivalent to (1). Hence, as stated in the introduction, there exists a fundamental system of solutions of (1)

$$y_1^{(j)} = \lambda_{1j}(x)x^{\alpha_j}, \quad y_2^{(j)} = \lambda_{2j}(x)x^{\alpha_j}, \quad \cdots, \quad y_n^{(j)} = \lambda_{nj}(x)x^{\alpha_j} \quad (j=1, 2, \cdots, n),$$

where $\lambda_{ij}(x)$ are analytic at $x=\infty$ or have a pole at that point. As there are *precisely* n formal solutions,

$$y_1 = x^{\alpha_j}A_{1j}(x), \quad y_2 = x^{\alpha_j}A_{2j}(x), \quad \cdots, \quad y_n = x^{\alpha}A_{nj}(x), \quad (j=1, 2, \cdots, n),$$

$$A_{ij}(x) = A_{ij} + A_{ij}^{(1)}\frac{1}{x} + \cdots,$$

* For an analogous reduction in the case $q=0$ see SCHLESINGER, loc. cit., pp. 183, 184.

we infer that $\lambda_{ij}(x) \equiv A_{ij}(x)$ and is therefore analytic at $x = \infty$. This is the fundamental result for a regular singular point and is due to SAUVAGE.*

§ 4. *Generalized Laplace integrals.*

The problem of the paper now resolves itself into a study of the solutions of (32) for $q \geqq 0$.

Let us consider first the case $q = 0$. The equations (32) are then

$$
\begin{aligned}
x\frac{dY_1}{dx} &= (p_{11}^{(0)} + \alpha_1 x)Y_1 + p_{12}^{(0)}Y_2 + \cdots + p_{1n}^{(0)}Y_n,\\
x\frac{dY_2}{dx} &= p_{21}^{(0)}Y_1 + (p_{22}^{(0)} + \alpha_2 x)Y_2 + \cdots + p_{2n}^{(0)}Y_n, \qquad (33)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
x\frac{dY_n}{dx} &= p_{n1}^{(0)}Y_1 + p_{n2}^{(0)}Y_2 + \cdots + (p_{nn}^{(0)} + \alpha_n x)Y_n.
\end{aligned}
$$

The formal solutions will have the form $(T_1^{(j)}, T_2^{(j)}, \cdots, T_n^{(j)})$ $(j = 1, 2, \cdots, n)$, where

$$
\begin{aligned}
T_i^{(j)} &= e^{\alpha_j x} x^{l_j} B_{ij}(x),\\
B_{ij}(x) &= B_{ij} + B_{ij}^{(1)}\frac{1}{x} + \cdots.
\end{aligned}
$$

Direct substitution shows that

$$l_j = p_{jj}^{(0)} \qquad (j = 1, 2, \cdots, n)$$

In this system of linear differential equations write

$$Y_i = \int e^{xz}\eta_i(z)\,dz \qquad (i = 1, 2, \cdots, n). \tag{34}$$

By employing the formula

$$\int e^{xz} x\,u(z)\,dz = [e^{xz}u(z)] - \int e^{xz}\,du(z),$$

in which the path of integration will be subsequently so chosen that the bracketed term may be omitted, we obtain, by equating the coefficients of e^{xz} in the integrands on both sides of (33),

$$
\begin{aligned}
-(z-\alpha_1)\frac{d\eta_1}{dz} &= (p_{11}^{(0)} + 1)\eta_1 + p_{12}^{(0)}\eta_2 + \cdots + p_{1n}^{(0)}\eta_n,\\
-(z-\alpha_2)\frac{d\eta_2}{dz} &= p_{21}^{(0)}\eta_1 + (p_{22}^{(0)} + 1)\eta_2 + \cdots + p_{2n}^{(0)}\eta_n, \qquad (35)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
-(z-\alpha_n)\frac{d\eta_n}{dz} &= p_{n1}^{(0)}\eta_1 + (p)_{n2}^{(0)}\eta_2 + \cdots + (p_{nn}^{(0)} + 1)\eta_n.
\end{aligned}
$$

* Annales de l'École Normale, series 3, vol. 3 (1886), p. 392. The analogous result for a single equation of the nth order is due to FUCHS.

We shall show how to employ (35), thus formally deduced, to obtain an actual representation (34) of the solutions.

The system (35) has $n+1$ regular singular points at $z=\alpha_1$, $z=\alpha_2$, $\cdots$, $z=\alpha_n$ and $z=\infty$, and no other singular points. The exponents at $x=\alpha_j$ are all zero except one which has the value $-p_{jj}^{(0)}-1=-l_j-1$. These results show that we may choose a fundamental system of n solutions, of which $n-1$ are regular at $x=\alpha_j$ and one has the form

$$\eta_1^{(j)}=(z-\alpha_j)^{-l_j-1}\mathfrak{P}_1^{(j)}(z-\alpha_j), \quad \eta_2^{(j)}=(x-\alpha_j)^{-l_j-1}\mathfrak{P}_2(z-\alpha_j).$$

$$\cdots, \quad \eta_n^{(j)}=(z-\alpha_j)^{-l_j-1}\mathfrak{P}_n(z-\alpha_j)$$

in which $\mathfrak{P}_i^{(j)}(z-\alpha_j)$ is regular at $z=\alpha_j$. This is true except possibly when l_j is an integer. We will refer to this possibility later.

The formula (34) will represent a solution of the given canonical system of equations provided we write

$$\eta_1=\eta_1^{(j)}, \quad \eta_2=\eta_2^{(j)}, \quad \cdots, \quad \eta_n=\eta_n^{(j)},$$

and the corresponding contour c_j of integration is taken to be a "*loop circuit*" that is, a circuit extending along a (properly chosen) ray from $z=\infty$ to the vicinity of $z=\alpha_j$, and then around the point in a negative sense, returning finally to $z=\infty$ along the ray. The direction of the ray must be so chosen that it does not contain another singular point $z=\alpha_k$, and that $R(xz)$ (the real part of xz) is negative along the ray. When this is done there is no difficulty in proving that each set of formulas

$$Y_i^{(j)}=\int_{c_j}e^{xz}(z-\alpha_j)^{-l_j-1}\mathfrak{P}_i^{(j)}(z-\alpha_j)\,dz \qquad (i=1, 2, \cdots, n)$$

represents a solution of (33), at least in certain sectors of the plane.

In the case in which $-l_j-1$ is a positive integer the loop circuit is replaced by a straight line path from $z=0$ to $z=\infty$. When $-l_j-1$ is negative or zero, there will be a solution which contains a logarithm. We choose this solution to replace $\eta_1^{(j)}, \eta_2^{(j)}, \cdots, \eta_n^{(j)}$. In all cases we obtain a formula for each singular point.

This elegant method of attack (for a single linear differential equation of the nth order with rational coefficients) was given by POINCARÉ in a fundamental paper.* The method has been more fully developed by HORN,† who shows that a direct study of the solutions can be made by these formulas in the complete vicinity of $x=\infty$.

When $q>0$ the above method fails. POINCARÉ has indicated a method of

*American Journal of Mathematics, vol. 7 (1885), pp. 217–232.

†Mathematische Annalen, vol. 50 (1897), pp. 525–556.

quadratures reducing the case $q > 0$ to the case $q = 0$, but the application of this method seems impracticable.*

It is the aim of this paragraph to show that, at least in the case before us, formulas exist for $q > 0$, analogous to those given when $q = 0$.†

First let us consider the case $q = 1$. The equations (32) are in this case

$$
\begin{aligned}
x\frac{dY_1}{dx} &= (p_{11}^{(0)} + p_{11}^{(1)}x + \alpha_1 x^2)Y_1 + (p_{12}^{(0)} + p_{12}^{(1)}x)Y_2 + \cdots + (p_{1n}^{(0)} + p_{1n}^{(1)}x)Y_n, \\
x\frac{dY_2}{dx} &= (p_{21}^{(0)} + p_{21}^{(1)}x)Y_1 + (p_{22}^{(0)} + p_{22}^{(1)}x + \alpha_2 x^2)Y_2 + \cdots + (p_{2n}^{(0)} + p_{2n}^{(1)}x)Y_n, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
x\frac{dY_n}{dx} &= (p_{n1}^{(0)} + p_{n1}^{(1)}x)Y_1 + (p_{n2}^{(0)} + p_{n2}^{(1)}x)Y_2 + \cdots + (p_{nn}^{(0)} + p_{nn}^{(1)}x + \alpha_n x^2)Y_n.
\end{aligned} \tag{36}
$$

The formal solutions are $(T_1^{(j)}, T_2^{(j)}, \cdots, T_n^{(j)})$ $(j = 1, 2, \cdots, n)$, where

$$
T_i^{(j)} = e^{\alpha_j \frac{x^2}{2} + \beta_j x} x^{l_j} B_{ij}(x),
$$

$$
B_{ij}(x) = B_{ij} + B_{ij}^{(1)}\frac{1}{x} + \cdots.
$$

If these formal expressions be substituted in (36), we obtain

$$
\beta_j = p_{jj}^{(1)} \qquad (j = 1, 2, \cdots, n). \tag{37}
$$

We shall take as the generalized form of (34) in this case

$$
Y_i = \int e^{xz}\,[\eta_{i0}(z) + x\eta_{i1}(z)]\,dz \qquad (i = 1, 2, \cdots, n). \tag{38}
$$

Now substitute this expression (38) in (36), integrating by parts according to the formula

$$
\int e^{xz} x^2 u(z)\,dz = [e^{xz}u(z)] - \int e^{xz}\,du(z),
$$

in order to reduce the terms which contain x^3 and x^2 to terms in x^1 and x^0 respectively. If the bracketed terms are omitted and the coefficients of e^{xz} and xe^{xz} in the integrands are then equated, there result the equations:

$$
\begin{aligned}
-(2z - \alpha_i)\frac{d\eta_{i1}}{dz} &= \eta_{i1} + \sum_{j=1}^{n}[p_{ij}^{(0)}\eta_{j1} + p_{ij}^{(1)}\eta_{j0}] \qquad (i = 1, 2, \cdots, n), \\
-(2z - \alpha_i)\frac{d\eta_{i0}}{dz} + \sum_{j=1}^{n} p_{ij}^{(1)}\frac{d\eta_{i1}}{dz} &= 2\eta_{i0} + \sum_{j=1}^{n} p_{ij}^{(0)}\eta_{j0} \quad (i = 1, 2, \cdots, n).
\end{aligned} \tag{39}
$$

* HORN, Acta Mathematica, vol. 23 (1900), pp. 177–201.

† Cf. CUNNINGHAM, Proceedings of the London Mathematical Society, ser. 2, vol. 4 (1906), pp. 374–383.

Thus we reach a system of $2n$ linear equations of the first order in z to determine the $2n$ functions

$$\eta_{11}, \eta_{21}, \cdots, \eta_{n1}, \eta_{10}, \eta_{20}, \cdots, \eta_{n0}.$$

This system of equations is analogous to (35). It has clearly singular points at $z = \alpha_1/2, z = \alpha_2/2, \cdots, z = \alpha_n/2$ and $z = \infty$, the last of which is a regular singular point. *The finite singular points are not in general regular*, although a slight consideration proves them to be of rank $q = 0$ at most.

It is possible however to make a given one of these singular points $z = \alpha_r/2$ a regular point by first making in (36) the simple preliminary transformation

$$Y_i = e^{-\beta_r x} Y_i.$$

The new system of equations ($\overline{36}$), and hence also ($\overline{39}$) will be the same as before except that $p_{ii}^{(1)}$ is replaced by $p_{ii}^{(1)} - \beta_r$ $(i = 1, 2, \cdots, n)$. *These n systems of equations* ($\overline{36}$), *in general distinct and obtained by taking* $r = 1, 2, \cdots, n$, *replace the single system* (35) *of the case* $q = 0$.

The rth system of equations ($\overline{39}$) *has a regular singular point at* $z = \alpha_r/2$. For the system (39) may be written

$$\frac{d\eta_{i1}}{dz} = \frac{-1}{2z - \alpha_i}\left\{\eta_{i1} + \sum_{j=1}^{n}[p_{ij}^{(0)}\eta_{j1} + p_{ij}^{(1)}\eta_{j0}]\right\} \qquad (i = 1, 2, \cdots, n),$$

$$\frac{d\eta_{i0}}{dz} = \frac{-1}{2z - \alpha_i}\left\{2\eta_{i0} + \sum_{j=1}^{n} p_{ij}^{(0)}\eta_{j0} + \sum_{j=1}^{n}\frac{p_{ij}^{(1)}}{2z - \alpha_j}\left[\eta_{j1} + \sum_{k=1}^{n}(p_{jk}^{(0)}\eta_{k1} + p_{jk}^{(1)}\eta_{k0})\right]\right\}$$
$$(i = 1, 2, \cdots, n).$$

A sufficient condition that $z = \alpha_r/2$ be regular is that there is no pole at $z = \alpha_r/2$ of the second or higher order in the coefficients of $\eta_{11}, \cdots, \eta_{n1}, \eta_{10}, \cdots, \eta_{n0}$ in the right-hand members. Clearly such a pole cannot exist except in the second n of these equations when $i = r$, and then only if $p_{r,r}^{(1)} \neq 0$. But in ($\overline{39}$) $p_{rr}^{(1)}$ is replaced by $p_{rr}^{(1)} - \beta_r$, which vanishes by (37). Hence the statement is proved.

The exponents of the regular singular point $z = \alpha_r/2$ can be verified directly to be all 0 except two, which are

$$\frac{-l_r - 1}{2}, \quad \frac{-l_r - 2}{2}.$$

Hence there exists a solution $\eta_{11}^{(r)}, \cdots, \eta_{n1}^{(r)}, \eta_{10}^{(r)}, \cdots, \eta_{n0}^{(r)}$ of the form

$$\eta_{i0}^{(r)} = \left(z - \frac{\alpha_r}{2}\right)^{\frac{-l_r-1}{2}} \mathfrak{P}_{i0}^{(r)}\left(z - \frac{\alpha_r}{2}\right), \qquad \eta_{i1}^{(r)} = \left(z - \frac{\alpha_r}{2}\right)^{\frac{-l_r-1}{2}} \mathfrak{P}_{i1}^{(r)}\left(z - \frac{\alpha_r}{2}\right),$$

where $\mathfrak{P}_{i0}^{(r)}(z - \alpha_r/2)$ and $\mathfrak{P}_{i1}^{(r)}(z - \alpha_r/2)$ are analytic at $z = \alpha_r/2$. Each set of formulas

$$Y_i^{(j)} = e^{-\beta_j x} \int_{c_j} e^{x^q z} \left(z - \frac{\alpha_j}{2}\right)^{\frac{-l_j-1}{2}} \left[\mathfrak{P}_{i0}^{(j)}\left(z - \frac{\alpha_j}{2}\right) + x \mathfrak{P}_{i1}^{(j)}\left(z - \frac{\alpha_j}{2}\right)\right] dz$$
$$(i = 1, 2, \cdots, n),$$

will then represent a solution provided that c_j is a loop circuit to $z = \alpha_j/2$ so chosen that $R[x^q(z - \alpha_j/2)]$ is negative along the ray.

If $(-l_j - 1)/2$ is an integer, we can obtain solutions corresponding to the other exponent $(-l_j - 2)/2$ which will not be an integer.

The proof of the validity of the above formulas is exactly like that in the case $q = 0$.

Let us now consider the general case when we have

$$T_i^{(j)} = e^{\alpha_j \frac{z^{q+1}}{q+1} + \beta_j \frac{z^q}{q} + \cdots + \lambda_j z} x^{lj} B_{ij}(x),$$

$$B_{ij}(x) = B_{ij} + B_{ij}^{(1)} \frac{1}{x} + \cdots.$$

We write

$$Y_i = \int e^{xz^{q+1}z} \{ \eta_{i0}(z) + x\eta_{i1}(z) + \cdots + x^q \eta_{iq}(z) \} dz,$$

where the $\eta_{ij}(z)$ are functions to be determined. First we have by substitution in (32)

$$\int e^{xz^{q+1}z} \sum_{j=0}^{q} x^j [(q+1)x^{q+1}z + j] \eta_{ij}(z) dz = \sum_{j=1}^{n} \int e^{xz^{q+1}z} \left(\sum_{k=1}^{q} p_{ij}^{(k)} x^k \right) \left(\sum_{l=1}^{q} x^l \eta_{jl}(z) \right) dz$$
$$+ \int e^{xz^{q+1}} \alpha_i x^{q+1} \sum_{l=0}^{q} x^l \eta_{il}(z) dz \qquad (i = 1, 2, \cdots, n).$$

The terms involve $x^0, x, \cdots, x^{2q+1}$. We propose to satisfy each of the equations above by deleting in turn all but the terms in

$$x^s \text{ and } x^{q+1+s} \qquad (s = 0, 1, \cdots, q)$$

and then satifying each of $q + 1$ equations resulting. One obtains after the omission of a factor x^s

$$\int e^{xz^{q+1}z} [(q+1)x^{q+1}z + j] \eta_{is}(z) dz$$
$$= \sum_{j=1}^{n} \int e^{xz^{q+1}z} \left(\sum_{k+l=s} p_{ij}^{(k)} \eta_{jl}(z) + x^{q+1} \sum_{k+l=q+s+1} p_{ij}^{(k)} \eta_{jl}(z) \right) dz + \int e^{xz^{q+1}} \alpha_i x^{q+1} \eta_{is}(z) dz.$$

By an integration by parts

$$\int e^{xz^{q+1}z} x^{q+1} w(z) dz = [e^{xz^{q+1}z} w(z)] - \int e^{xz^{q+1}z} dw(z),$$

in which the bracketed terms are again cancelled by a subsequent choice of limits, we may transform the coefficient $e^{xz^{q+1}z}$ in the integrands so that it is inde-

pendent of x. Equating these coefficients we then obtain in a formal manner the system of $n(q+1)$ differential equations for the functions η_{is},

$$-[(q+1)z-\alpha_i]\frac{d\eta_{is}}{dz}+\sum_{j=1}^{n}\sum_{k+l=q+s+1}p_{ij}^{(k)}\frac{d}{dz}\eta_{jl}=(q+1-s)\eta_{is}+\sum_{j=1}^{n}\sum_{k+l=s}p_{ij}^{(k)}\eta_{jl}$$
$$(i=1,2,\cdots,n;\ s=0,1,\cdots,q).$$

These equations written in succession for $s=q, q-1, \cdots, 0$ are

$$-[(q+1)z-\alpha_i]\frac{d\eta_{iq}}{dz}=\eta_{iq}+\sum_{j=1}^{n}\sum_{k+l=q}p_{ij}^{(k)}\eta_{jl} \qquad (i=1,2,\cdots,n),$$

$$-[(q+1)z-\alpha_i]\frac{d\eta_{i,q-1}}{dz}+\sum_{j=1}^{n}p_{ij}^{(q)}\frac{d\eta_{jq}}{dz}=2\eta_{iq-1}+\sum_{j=1}^{n}\sum_{k+l=q-1}p_{ij}^{(k)}\eta_{jl}$$
$$(i=1,2,\cdots,n),$$

(40) . . . . . .

$$-[(q+1)z-\alpha_i]\frac{d\eta_{i0}}{dz}+\sum_{j=1}^{n}p_{ij}^{(q)}\frac{d\eta_{j1}}{dz}+\cdots+\sum_{j=1}^{n}p_{ij}^{(0)}\frac{d\eta_{j,q+1}}{dz}$$
$$=(q+1)\eta_{i0}+\sum_{j=1}^{n}\sum_{k+l=0}p_{ij}^{(k)}\eta_{jl} \qquad (i=1,2,\cdots,n).$$

The singular points of (40) are at once seen to be $z=\alpha_1/(q+1)$, $z=\alpha_2/(q+1)$, $\cdots$, $z=\alpha_n/(q+1)$ and $z=\infty$, the last of which is clearly a regular singular point while the others are of rank q at most.

As in the case $q=1$ we can, however, make a given one of the points $z=\alpha_r/(q+1)$ a regular singular point by making a preliminary transformation

$$\overline{Y}_i=e^{-\beta_r\frac{x^q}{q}-\gamma_r\frac{x^{q-1}}{q-1}\cdots-\lambda_r x}Y_i.$$

The new system ($\overline{32}$), and hence also ($\overline{40}$), will not be changed except that

$$p_{ii}^{(q)},\ p_{ii}^{(q-1)},\ \cdots,\ p_{ii}^{(0)} \qquad (i=1,2,\cdots,n),$$

will be replaced by

$$p_{ii}^{(q)}-\beta_r,\ p_{ii}^{(q-1)}-\gamma_r,\ \cdots,\ p_{ii}^{(0)}-\lambda_r$$

respectively.

These n systems of equations ($\overline{40}$) ($r=1,2,\cdots,n$) *will, as in the cases* $q=0$ *and* $q=1$, *lead to Laplace integral formulas.*

We shall now prove that *the system* (40) *has a regular singular point at* $z=\alpha_r/(q+1)$. To prove this fact we note that, in view of the preliminary transformation assumed to have been made, the system (32) admits a formal solution $T_1^{(r)}, T_2^{(r)}, \cdots, T_n^{(r)}$, where

$$T_i^{(r)}=e^{\alpha_r\frac{x^{q+1}}{q+1}}x^{l_r}B_{ir}(x) \qquad (i=1,2,\cdots,n).$$

Now it is easily proved that if the system (32) has a formal solution of this kind, then the adjoint system of equations,

$$\frac{dz_j}{dx} = -\sum_{i=1}^{n} z_i P_{ij}(x),$$

will have a formal solution

$$z_j(x) = e^{-\alpha_r \frac{x^{q+1}}{q+1}} x^{-l_r} \{ C_j + C_j^{(1)} \frac{1}{x} + \cdots \} \qquad (j=1, 2, \cdots, n),$$

in which not every C_j $(j=1, 2, \cdots, n)$ is zero. If this formal solution be substituted in the adjoint system, we obtain the following set of linear equations:

$$\alpha_r C_j = \alpha_j C_j \qquad (j=1, 2, \cdots, n),$$

$$\alpha_r C_j^{(1)} = \alpha_j C_j^{(1)} + \sum_{i=1}^{n} C_i p_{ij}^{(q)} \qquad (j=1, 2, \cdots, n),$$

. . . . . .

$$\alpha_r C_j^{(q)} = \alpha_j C_j^{(q)} + \sum_{i=1}^{n} (C_i^{(q-1)} p_{ij}^{(q)} + \cdots + C_i p_{ij}^{(1)}) \qquad (j=1, 2, \cdots, n).$$

From the first n of the above equations we infer that

$$C_j = 0 \quad (j \neq r), \qquad C_r \neq 0.$$

Multiply now the n equations of the $(k+1)$th line in (40) by $C_i (i=1, 2, \cdots, n)$ respectively, the kth set by $C_i^{(1)}$ $(i=1, 2, \cdots, n)$, and so on. Then add the first nk equations thus obtained. In view of the above relations between the $B_i^{(j)}$ one finds

$$-[(q+1)z-\alpha_r]\sum_{i=1}^{n}\left\{ C_i \frac{d\eta_{i,\,q-k}}{dz} + C_i^{(1)} \frac{d\eta_{i,\,q-k+1}}{dz} + \cdots + C_i^{(k)} \frac{d\eta_{iq}}{dz} \right\} = L_k(\eta),$$

in which $L_k(\eta)$ denotes a certain expression linear in η_{ij} with constant coefficients. Since $C_i = 0$ $(i \neq r)$, $C_r \neq 0$, we may write this in the form

$$(41) \quad \frac{d\eta_{r,\,q-k}}{dz} = -\frac{1}{C_r}\sum_{i=1}^{n}\left\{ C_i^{(1)} \frac{d}{dz}\eta_{i,\,q-k+1} + \cdots + C_i^{(k)} \frac{d}{dz}\eta_{iq} \right\} + \frac{1}{(q+1)z-\alpha_r} L_k(\eta)$$

$$(k=0, 1, 2, \cdots, q).$$

We are now able to prove that $z = \alpha_r/(q+1)$ is a regular singular point. This will be proved if we show that when we solve (40) for the derivatives of η_{ij} in terms of the η_{ij} themselves, the coefficients of the expressions obtained have at $z = \alpha_r/(q+1)$ poles of the first order at highest. The expressions for $d\eta_{iq}/dz$ $(i=1, 2, \cdots, n)$ certainly satisfy this condition, as the first set of equations (40) show. In solving for $d\eta_{i,\,q-1}/dz$ $(i=1, 2, \cdots, n)$ we distinguish the cases $i \neq r$ and $i = r$. Since in the first case $d\eta_{i,\,q-1}/dz$ appears with a coefficient $-[(q+1)z-\alpha_i]$ not zero at $z = \alpha_r/(q+1)$, it is clear that poles of

higher than the first order do not appear. To solve for $d\eta_{r,q-1}/dz$ we use equation (41) for $k=1$ and deduce the same fact. In this way we solve for $d\eta_{iq}/dz$, $d\eta_{i,q-1}/dz$, $\cdots$, $d\eta_{i0}/dz$, each time making use of the appropriate equation (41), to prove that no poles of higher than the first order arise. Thus the statement is finally proved.

The $n(q+1)$ exponents at $z=\alpha_r/(q+1)$ are all zero excepting $q+1$ of them, which are $(-l_r-1)/(q+1)$, $(-l_r-2)/(q+1)$, $\cdots$, $(-l_r-q-1)/(q+1)$. I have not found a direct proof of this fact, such as no doubt exists. An outline of an indirect proof is found in the foot-note p. 462.

We will have then a solution $\eta_{ij}^{(r)}$ $(i=1,2,\cdots,n;\ j=0,1,\cdots,q)$ such that

$$\eta_{ij}^{(r)}=\left(z-\frac{\alpha_r}{q+1}\right)^{\frac{-l_r-1}{q+1}}\mathfrak{P}_{ij}^{(r)}\left(z-\frac{\alpha_r}{q+1}\right),$$

where $\mathfrak{P}_{ij}^{(r)}[z-\alpha_r/(q+1)]$ is a function analytic at $z=\alpha_r/(q+1)$. This leads at once to the formulas

$$(42)\quad Y_i^{(j)}=e^{\beta_j\frac{x^{q+1}}{q}+\cdots+\lambda_j x}\int_{c_j}e^{xz^{q+1}z}\left(z-\frac{\alpha_j}{q+1}\right)^{\frac{-l_j-1}{q+1}}\sum_{k=0}^{q}x^k\mathfrak{P}_{ik}^{(j)}\left(z-\frac{\alpha_r}{q+1}\right)dz$$

where c_j is a loop circuit about $z=\alpha_j/(q+1)$ so chosen that $R\{x^{q+1}[z-\alpha_j/(q+1)]\}$ is negative along the ray. These actually represent solutions, as can be proved just as in the cases $q=0$ and $q=1$.

If $(-l_j-1)/(q+1)$ is an integer, one may use solutions corresponding to the remaining non-zero exponents, none of which are integers.

§ 5. *Asymptotic representation.*

If one makes the substitution

$$w=x^{q+1}\left(z-\frac{\alpha_r}{q+1}\right)$$

in the formulas (42) for $j=r$, there is obtained the expression

$$Y_i^{(r)}=e^{P_r(x)}x^{l_r}\int_{c_r}e^{w}w^{\frac{-l_r-1}{q+1}}\sum_{j=0}^{q}x^j\mathfrak{P}_{ij}^{(r)}(x^{-q-1}w)\,dw,$$

where c_r' is a loop circuit to $w=0$. When one expands $\mathfrak{P}_{ij}(x^{-q-1}w)$ and evaluates the successive terms by means of the formula

$$\int_{c_r'}e^{w}w^{\kappa}\,dw=(e^{-\pi\kappa\sqrt{-1}}-e^{\pi\kappa\sqrt{-1}})\,\Gamma(\kappa+1),$$

there results for $Y_i^{(j)}$ a sum of terms of the same form as in the expression $T_i^{(j)}$ plus terms which approach zero as x becomes infinite.*

* For a more complete consideration of a like point see the paper by HORN previously referred to, Mathematische Annalen, vol. 50 (1897), pp. 527–531.

In order to state the result concisely we shall modify slightly the conventional definition of asymptotic representation.

DEFINITION. We may say that Y_i *is asymptotically represented by* $T_i^{(r)}$ *along the ray* $\arg x = \phi$, or in symbols

$$Y_i \sim T_i^{(r)}, \qquad \arg x = \phi,$$

if for some sector including $\arg x = \phi$ *as an interior ray and for all values of* m,

$$Y_i = e^{P_r(x)} x^{k_r} \left\{ A_{ir} + \frac{A_{ir}^{(1)}}{x} + \cdots + \frac{A_{ir}^{(m-1)}}{x^{m-1}} + \frac{\epsilon}{x^m} \right\},$$

in which $\lim \epsilon = 0$ *as* x *becomes infinite.*

The result is then that we have

$$Y_i^{(j)} \sim T_i^{(j)}$$

along any ray for which $R\{x^{q+1}[z - \alpha_j/(q+1)]\}$ is negative. These expressions $T_i^{(j)}$ are, of course, identical with the $T_i^{(j)}$ which occur in the formal solutions of (32).*

The ray of the loop circuit to $z = \alpha_j/(q+1)$ may vary in direction as long as it does not intersect one of the remaining $n-1$ finite singular points $z = \alpha_l/(q+1)$. It is a simple matter to give the actual sectors on which in general the above integral formulas hold. Denote by $\tau_1, \tau_2, \cdots, \tau_N$ the $N = n(n-1)(q+1)$ rays in angular order for which

$$R\{(\alpha_j - \alpha_k)x^{q+1}\} = 0 \qquad (j \neq k),$$

and also write $\tau_{N+1} = \tau_1 + 2\pi$. *We assume these rays to be distinct.* Furthermore, the real part of some particular one of the differences $(\alpha_j - \alpha_k)x^{q+1}$ changes from positive to negative as we pass from the sector (τ_{m-1}, τ_m) to (τ_m, τ_{m+1}). Denote the subscripts in this difference by j_m and k_m. The ray of the loop circuit c_j has an argument intermediate between a certain successive pair of arguments

$$\arg(\alpha_{j_{m'}} - \alpha_{k_{m'}}) \text{ and } \arg(\alpha_{j_m} - \alpha_{k_m})$$

for which $k_{m'} = k_m = j$. In this case $R\{x^{q+1}[z - \alpha_j/(q+1)]\}$ remains negative for

$$\tau_m < \arg x < \tau_{m'} + \frac{\pi}{q+1}.$$

The formula (42)

$$Y_1 = Y_1^{(j)}, \quad Y_2 = Y_2^{(j)}, \quad \cdots, \quad Y_n = Y_n^{(j)}$$

* At this point it is easy to see that the exponents at $z = \alpha_r/(q+1)$ have the values above stated. For by comparison of the Laplace integrals and the formal solutions we get these values as possible exponents, and no others. But in the special case for which $P_{ij}(x) = 0$, $P_{ii} = p_{ii}^{(0)} + \alpha_i x^{q+1}$, it may be at once proved that the exponents have precisely these values. Then if the coefficients of the polynomials are changed continuously, the exponents are also so changed and hence retain the form specified.

represents a set of $q+1$ solutions fixed by assigning the sector in which the ray of the loop circuit is to lie. This formula is valid for any one of the $q+1$ corresponding sectors

$$\tau_m < arg\, x < \tau_{m'} + \frac{\pi}{q+1}.$$

It is to be noted that m and m' are *any* pair of successive values of m for which $k_m = k_{m'} = j$.

If now we pass back to the solutions of (1), we infer that *there exist solutions $y_1, y_2, \cdots, y_n$ of (1) such that*

$$y_i \sim S_i^{(j)} \qquad (i = 1, 2, \cdots, n)$$

within any given sector

$$\tau_m < arg\, x < \tau_{m'} + \frac{\pi}{q+1}.$$

This result has also been derived in certain special cases by HORN,* who has used the method of successive approximation.

§ 6. *Characterization at $z = \infty$.*

At this point we can readily give a characterization of the nature of the solutions of (32) and thence of the system (1) equivalent to it at $x = \infty$.

THEOREM. *There exist $N = n(n-1)(q+1)$ fundamental systems of solutions of the canonical system* (32),

$$\begin{matrix} Y_{1m}^{(1)}, & Y_{2m}^{(1)}, & \cdots, & Y_{nm}^{(1)}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & & \\ Y_{1m}^{(n)}, & Y_{2m}^{(n)}, & \cdots, & Y_{nm}^{(n)}, \end{matrix} \qquad (m = 1, 2, \cdots, N), \tag{43}$$

such that

$$Y_{im}^{(j)} \sim T_i^{(j)}, \qquad \tau_m \leqq \arg x < \tau_{m+1},$$

and such that

$$Y_{i,\,m+1}^{(j)} = Y_{im}^{(j)} \qquad (j \neq j_m)$$

$$Y_{i,\,m+1}^{(j_m)} = Y_{im}^{(j_m)} + A_m Y_{im}^{(k_m)},$$

provided we write $Y_{i,\,N+1}^{(j)} = e^{2\pi l_j \sqrt{-1}} Y_{i1}^{(j)}$.

Proof. In order to avoid confusion we will state at the outset that the solutions (42) are not in general the solutions referred to in this theorem. The latter retain their asymptotic form throughout minimum sectors, while the former retain it throughout maximum sectors.

*Crelle's Journal, vol. 133 (1907), pp. 19–67

An immediate conclusion of the results of the preceding paragraph is *that there exists for every ray arg* $x=\phi$ *a set of solutions* $Y_1^{(j)}, Y_2^{(j)}, \cdots, Y_n^{(j)}$ $(j=1, 2, \cdots, n)$ *such that along this ray*

$$Y_i^{(j)} \sim T_i^{(j)}.$$

In virtue of this fact it is possible to separate the x-plane into some finite number of closed sectors σ, in each of which there exists a fundamental system of solutions $Y_1^{(j)}, Y_2^{(j)}, \cdots, Y_n^{(j)}$ $(j=1, 2, \cdots, n)$ such that $Y_i^{(j)} \sim T_i^{(j)}$ for x in σ. We choose these sectors so that the rays arg $x=\tau_m$ are included at most one within each sector σ.

Throughout a sector σ not containing an inner ray τ_i *every solution* $Y_1, Y_2, \cdots, Y_n$ of (32) has a definite asymptotic form. For the general solution of (32) is

$$Y_i = c_1 Y_i^{(1)} + c_2 Y_i^{(2)} + \cdots + c_n Y_i^{(n)} \qquad (i=1, 2, \cdots, n), \tag{44}$$

which gives the asymptotic relationship

$$Y_i \sim c_1 T_i^{(1)} + c_2 T_i^{(2)} + \cdots + c_n T_i^{(n)}.$$

The relative magnitudes of these n terms as x becomes infinite in σ clearly depend on the relative magnitude of

$$R(\alpha_1 x^{q+1}), \quad R(\alpha_2 x^{q+1}), \quad \cdots, \quad R(\alpha_n x^{q+1}). \tag{45}$$

This does not change except at the rays arg $x=\tau_1, \tau_2, \cdots, \tau_N$. Suppose for a moment that indices 1 to n are so chosen that the quantities (45) appear in order of descending magnitude. If c_k is the first of the quantities $c_1, c_2, \cdots, c_n$ not zero, one has clearly for x in σ

$$Y_i \sim c_k T_i^{(k)}.$$

Furthermore, since the successive regions σ have a ray in common, every solution $Y_1, Y_2, \cdots, Y_n$ preserves its asymptotic form throughout successive sectors σ until one which contains a ray $\tau_1, \tau_2, \cdots, \tau_N$ is reached. We may therefore unite into a single sector σ_m the region σ which contain a given ray τ_m, and all those which succeed it in angular order up to the one which includes the ray τ_{m+1}, and affirm that on each of these sectors σ_m there exists a fundamental system $Y_1^{(j)}, Y_2^{(j)}, \cdots, Y_n^{(j)}$ $(j=1, 2, \cdots, n)$, for which $Y_i^{(j)} \sim T_i^{(j)}$ throughout the sector.

Next let us determine the character of the general solution $Y_1, Y_2, \cdots, Y_n$ in one of these sectors σ_m which includes within it the ray arg $x=\tau_m$. Here again the relative magnitude of the quantities (45) determines the important term in (44). But $R\{P_{j_m}(x)\}$ and $R\{P_{k_m}(x)\}$ *change order of magnitude along arg* $x=\tau_m$, *and moreover*

$$R\{(\alpha_{j_m} - \alpha_{k_m})x^{q+1}\} > 0 \qquad (\arg x < \tau_m),$$

$$R\{(\alpha_{j_m} - \alpha_{k_m})x^{q+1}\} < 0 \qquad (\arg x > \tau_m).$$

If we suppose indices so chosen that the quantities (45) are in order of magnitude at the first boundary ray of σ_m, two successive terms of (45), say the sth and the $(s+1)$th, will change order of magnitude for $\arg x = \tau_m$. We see by (44) that the solution $Y_1, Y_2, \cdots, Y_n$ will preserve its asymptotic form in σ_m unless

$$c_1 = c_2 = \cdots = c_{s-1} = 0, \qquad c_s \neq 0, c_{s+1} \neq 0.$$

In this case the solution is of the asymptotic form

$$c_s T_1^{(s)}, c_s T_2^{(s)}, \cdots, c_s T_n^{(s)}$$

at the initial ray of the sector, and of the asymptotic form

$$c_{s+1} T_1^{(s+1)}, c_{s+1} T_2^{(s+1)}, \cdots, c_{s+1} T_n^{(s+1)}$$

at the terminal ray.

If, however, we have two solutions such that at the initial ray of the sector

$$\begin{aligned} Y_i &\sim c_s T_i^{(s)} \\ Y_i' &\sim c_{s+1}' T_i^{(s+1)} \end{aligned} \qquad (i = 1, 2, \cdots, n),$$

we can always choose a combination $Y_i'' = Y_i + A Y_i'$ so that $(Y_1'', Y_2'', \cdots, Y_n'')$ preserves its asymptotic form $c_s T_1^{(s)}, c_s T_2^{(s)}, \cdots, c_s T_n^{(s)}$ throughout the sector σ_m. For we have

$$\begin{aligned} Y_i &= c_s Y_i^{(s)} + c_{s+1} Y_i^{(s+1)} + \cdots + c_n Y_i^{(s)}, \\ Y_i' &= c_{s+1}' Y_i^{s+1} + \cdots + c_n' Y_i^{(n)}. \end{aligned}$$

The required value of A is $-c_{s+1}/c_{s+1}'$.

Let now $Y_{11}^{(j)}, Y_{21}^{(j)}, \cdots, Y_{n1}^{(j)}$ $(j = 1, 2, \cdots, n)$ be any fundamental system of solutions of (32) such that

$$Y_{i1}^{(j)} \sim T_i^{(j)}$$

for x on σ_1. All these solutions will preserve their asymptotic form throughout σ_2 except perhaps the solution

$$Y_{11}^{(j_1)}, Y_{21}^{(j_1)}, \cdots, Y_{n1}^{(j_1)}$$

in view of the preceding discussion. However, in this case we can choose a constant A, so that the solution

$$Y_{11}^{(j_1)} + A_1 Y_{11}^{(k_1)}, Y_{21}^{(j_1)} + A_1 Y_{21}^{(k_1)}, \cdots, Y_{n1}^{(j_1)} + A_1 Y_{n1}^{(k_1)}$$

preserves its asymptotic form

$$T_1^{(j_1)}, T_2^{(j_1)}, \cdots, T_n^{(j_1)}$$

throughout the sector σ_2. Therefore if we write

$$\begin{aligned} Y_{12}^{(j)} &= Y_{11}^{(j)} \qquad (j \neq j_1), \\ Y_{12}^{(j_1)} &= Y_{i1}^{(j_1)} + A_1 Y_{i1}^{(k_1)} \end{aligned}$$

the new fundamental system of solutions $Y_{12}^{(j)}, Y_{22}^{(j)}, \cdots, Y_{n2}^{(j)}$ $(j=1, 2, \cdots, n)$ preserves its asymptotic form throughout the sector σ_2.

Passing now to the sector σ_3, we obtain a third fundamental system of solutions, and so on. Thus we define $Y_{i2}^{(j)}, Y_{i3}^{(j)}, \cdots, Y_{i,N+1}^{(j)}$ successively in terms of $Y_{i1}^{(j)}$.

These fundamental systems of solutions have therefore the desired properties stated in the theorem except that we need not have

$$Y_{i,N+1}^{(j)} = e^{2\pi i\rho^{\gamma}\sqrt{-1}}\, Y_{i1}^{(j)} \qquad (i, j=1, 2, \cdots, n).$$

These relations, too, will be satisfied when a proper choice of the initial fundamental system

$$Y_{11}^{(j)}, \ Y_{21}^{(j)}, \ \cdots, \ Y_{n1}^{(j)} \qquad (j=1, 2, \cdots, n) \tag{46}$$

is made. To prove this fact we show first that *the final fundamental system of solutions*

$$Y_{1,N+1}^{(j)}, \ Y_{2,N+1}^{(j)}, \ \cdots, \ Y_{n,N+1}^{(j)} \qquad (j=1, 2, \cdots, n) \tag{47}$$

is entirely independent of the choice of the initial system (46).

Let

$$Z_{1m}^{(j)}, \ Z_{2m}^{(j)}, \ \cdots, \ Z_{nm}^{(j)} \qquad (j=1, 2, \cdots, n; \ m=1, 2, \cdots, N)$$

be a second set of solutions of the same kind; the constants which correspond to A_m will be denoted by B_m.

Let $R(\alpha_s x^{q+1})$ be that one of the quantities (45) which *is least in magnitude* in (τ_1, τ_2). There can only be one solution of (32) asymptotic to $T_1^{(s)}, T_2^{(s)}, \cdots, T_n^{(s)}$ on (τ_1, τ_2), and we must have

$$Y_{i1}^{(s)} = Z_{i1}^{(s)} \qquad (i=1, 2, \cdots, n) \tag{48}$$

in this sector. Now each pair of the quantities (45) become equal for $2(q+1)$ values of τ,

$$\tau = \tau', \ \tau' + \frac{\pi}{q+1}, \ \tau' + \frac{2\pi}{q+1}, \ \cdots, \ \tau' + \frac{2q-1}{q+1}\pi.$$

It is clear, then, that on the sector

$$\tau_1 \leqq \arg x < \tau_1 + \frac{\pi}{q+1} = \tau_\mu \qquad \left(\mu = \frac{n(n-1)}{2}\right), \tag{49}$$

each pair of these quantities (45) become equal once and only once. The quantity $R(\alpha_s x^{q+1})$ will therefore *increase over* the remaining quantities (45) as $\arg x$ changes from τ_1 to $\tau_1 + \pi/(q+1)$. It is therefore clear that

$$Y_{i1}^{(s)} = Y_{i2}^{(s)} = \cdots = Y_{i\mu}^{(s)},$$

$$Z_{i1}^{(s)} = Z_{i2}^{(s)} = \cdots = Z_{i\mu}^{(s)}.$$

Hence we find

$$Y^{(s)}_{im} = Z^{(s)}_{im} \qquad (i = 1, 2, \cdots, n)$$

for all values of m less than $\mu + 1$.

Next let $R(\alpha_t x^{q+1})$ be that one of the quantities (45) which is second in magnitude in (τ_1, τ_2). We must have in this case

$$Z^{(t)}_{im} = Y^{(t)}_{im} + cY^{(s)}_{im}$$

for $m = 1, 2, \cdots$, until a value $m = \theta$ is reached for which $R(\alpha_s x^{q+1})$ increases over $R(\alpha_t x^{q+1})$. It must be remembered that $R(\alpha_t x^{q+1})$ increases over all the quantities (45) except $R(\alpha_s x^{q+1})$ in (τ_1, τ_μ). Now in virtue of the relations

$$Y^{(t)}_{i,\theta+1} = Y^{(t)}_{i\theta} + A_\theta Y^{(s)}_{i\theta}, \qquad Y^{(s)}_{i,\theta+1} = Y^{(s)}_{i\theta},$$

$$Z^{(t)}_{i,\theta+1} = Z^{(t)}_{i\theta} + B_\theta Z^{(s)}_{i\theta}, \qquad Z^{(s)}_{i,\theta+1} = Z^{(s)}_{i\theta},$$

the preceding equality for $m = \theta$ may be written

$$Z^{(t)}_{i,\theta+1} - B_\theta Z^{(s)}_{i,\theta+1} = Y^{(t)}_{i,\theta+1} - A_\theta Y^{(s)}_{i,\theta+1} + cY^{(s)}_{i,\theta+1}. \tag{50}$$

Also since $R(\alpha_s x^{q+1})$ is greater than $R(\alpha_t x^{q+1})$ in $(\tau_\theta, \tau_{\theta+1})$ we conclude that

$$-B_\theta = -A_\theta + c.$$

We have already shown that

$$Y^{(s)}_{i,\theta+1} = Z^{(s)}_{i,\theta+1},$$

and we obtain from (50)

$$Y^{(t)}_{i,\theta+1} = Z^{(t)}_{i,\theta+1}.$$

Since for $m > \theta$, the quantity $R(\alpha_t x^{q+1})$ will increase over the remaining quantities (45), we infer that

$$Y^{(t)}_{im} = Z^{(t)}_{im} \qquad (i = 1, 2, \cdots n)$$

for all values of m greater than θ.

Likewise we have

$$Z^{(u)}_{i1} = Y^{(u)}_{i1} + cY^{(t)}_{i1} + dY^{(s)}_{i1},$$

where $R(\alpha_u x^{q+1})$ is the third of the quantities (45) in order of magnitude. The relation

$$Z^{(u)}_{im} = Y^{(u)}_{im} + cY^{(t)}_{im} + dY^{(s)}_{im}$$

will hold for successive values of m, the constants c and d only changing for a value of m such that one of the three quantities

$$R(\alpha_s x^{q+1}), \qquad R(\alpha_t x^{q+1}), \qquad R(\alpha_u x^{q+1}),$$

changes order of magnitude at $\arg x = \tau_m$. If the first increases over the second, the constant d may change; when the second increases over the third, the constant c becomes zero; when the first increases over the third, the con-

stant c becomes zero. From and after this fixed value of $m < \mu + 1$, for which the first increases over the third, we have

$$Y_{im}^{(u)} = Z_{im}^{(u)} \qquad (i = 1, 2, \cdots, n).$$

In this way we prove that after some fixed value of $m < \mu + 1$ we shall always have

$$Y_{im}^{(j)} = Z_{im}^{(j)} \qquad (i = 1, 2, \cdots, n),$$

whatever be the particular value of j. Therefore we have

$$Y_{im}^{(j)} = Z_{im}^{(j)} \qquad (i, j = 1, 2, \cdots, n)$$

for $m = \mu$, and of course for greater values of m, in particular $m = N + 1$.

Choose $Y_{i1}^{(j)}$ to be this fixed set $Y_{i,N+1}^{(j)}$ multiplied by $e^{-2\pi l_j \sqrt{-1}}$, and all the conditions of the theorem will be satisfied. This is a possible choice, since $T_i^{(j)}$ after a complete circuit of $x = \infty$ becomes $e^{2\pi l_j \sqrt{-1}} T_i^{(j)}$, and therefore we have $Y_{i,N+1}^{(j)} \sim e^{2\pi l_j \sqrt{-1}} T_i^{(j)}$ in (τ_1, τ_2). This completes the demonstration.

The theorem whose demonstration has just been completed leads at once to a like theorem concerning the solutions of (1).

THEOREM. *There exist $N = n(n-1)(q+1)$ fundamental solutions of* (1),

$$\begin{aligned} &y_{1m}^{(1)},\ y_{2m}^{(1)},\ \cdots,\ y_{nm}^{(1)}, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \qquad (m = 1, 2, \cdots, N+1), \\ &y_{1m}^{(n)},\ y_{2m}^{(n)},\ \cdots,\ y_{nm}^{(n)} \end{aligned}$$

such that

$$y_{im}^{(j)} \sim S_i^{(j)}, \qquad \tau_m \leqq arg\, x < \tau_{m+1},$$

and such that

$$y_{i,m+1}^{(j)} = y_{im}^{(j)} \qquad (j \neq j_m),$$

$$y_{im}^{(j_m)} = y_{im}^{(j_m)} + A_m y_{im}^{(k_m)},$$

provided we write $y_{i,N+1}^{(j)} = e^{2\pi l_j \sqrt{-1}} y_{i1}^{(j)}$.

This is a complete characterization of the solutions of (1) in the vicinity of $x = \infty$, since any system of functions $Y_{im}^{(j)}$ satisfying these conditions will form the solutions of a system of equations (1). The *characteristic constants* which determine the nature of the solutions are the $n(q+2)$ *exponential constants*

$$\alpha_j,\ \beta_j,\ \cdots,\ \lambda_j,\ \kappa_j \qquad (j = 1, 2, \cdots, n),$$

and the $n(n-1)(q+1)$ *transformation constants* $A_1, A_2, \cdots, A_N$. Not all of the constants A_m are independent, since instead of the particular n formal solutions used we might have multiplied each of these by an arbitrary constant. The new transformation constant would then be altered to

$$\bar{A}_m = \frac{c_{k_m}}{c_{j_m}} A_m.$$

Two sets of transformation constants thus related are considered to be the same. Any two systems (1) and ($\bar{1}$) with the same characteristic constants (except that κ_j and $\bar{\kappa}_j$ may differ by integers) may be proved to be equivalent at $x = \infty$, and conversely.

The total number of essential characteristic constants is accordingly

$$n(q+2)+n(n-1)(q+1)-(n-1)=n^2(q+1)+1.$$

But we can dispose of $n-1$ further constants of (32) by the transformation

$$\overline{Y}_i = c_i Y_i \qquad (i=1, 2, \cdots, n).$$

It is clear then that the canonical system of equations (32) also has at most $n^2(q+1)+1$ essential constants. If it be admitted that no further reduction of (32) is possible, we infer that *the characteristic constants are not connected by any necessary relation.*

§ 7. *Generalized Riemann problem.*

We now propose the following generalization of the Riemann problem: *To construct a system of n linear differential equations of the first order with prescribed singular points*

$$x_1, x_2, \cdots, x_m, x_{m+1}=\infty$$

of respective rank

$$q_1, q_2, \cdots, q_{m+1}$$

and with a given monodromic group, the characteristic constants being assigned for each singular point.

Consider the system of equations

$$\frac{dy_i}{dx}=\sum_{j=1}^{n}\left(\sum_{k=1}^{m}\sum_{l=1}^{q_k+2}\frac{A_{ijkl}}{(x-x_k)^l}+\sum_{k=0}^{q_{m+1}}B_{ijk}x^k\right)y_j \qquad (i=1, 2, \cdots, n), \tag{51}$$

with singular points $x_1, x_2, \cdots, x_m, x_{m+1}=\infty$ of prescribed rank $q_1, q_2, \cdots, q_{m+1}$. In this equation appear

$$n^2\left(\sum_{l=1}^{m+1}q_l+2m+1\right)$$

arbitrary constants.

In order to fix the problem we consider the fundamental system of solutions $(y_1^{(1)}, \cdots, y_n^{(1)}), \cdots, (y_n^{(1)}, \cdots, y_n^{(n)})$ such that $y_i^{(j)}$ at some point $x=a$ reduces to δ_{ij}, and take the group of this particular fundamental system of solutions to be assigned.

The monodromic group contains n^2m arbitrary constants, n^2 for each fundamental substitution. Furthermore, the characteristic constants are

$$\sum_{1}^{m+1}[n^2(q_l+1)+1]$$

in number; but between these constants and those of the group there are $n(m+1)$ relations, since the n exponents at each of the $m+1$ singular points are determined either by the group or by the characteristic constants.

We must further prescribe at each singular point precisely which solutions are to correspond to the canonical fundamental system of solutions given in the second theorem of § 6. Now the group itself determines the correspondence (since it determines the exponents) except for n multiplicative constants. There are accordingly

$$(n-1)(m+1)$$

further conditions. Thus there are in all

$$n^2 m + \sum_{l=1}^{m+1} [n^2(q_l+1)+1] - n(m+1) + (n-1)(m+1)$$

conditions to be satisfied.

The number of conditions equal the number of constants at our disposal. This is the basis on which we postulate the possibility of the general problem.

THE UNIVERSITY OF WISCONSIN.

AUTOMORPHISMS OF ORDER TWO*

BY

G. A. MILLER

§ 1. *Introduction.*

If an operator t transforms a group G into itself without being commutative with each operator of G, and if t^2 is commutative with each operator of G, then t is said to transform the operators of G according to an automorphism of order two. Moreover, every automorphism of G which is of order 2 may be obtained by transforming G by operators having the given properties of t.† Let s be any operator of G and assume that

$$t^{-1}st = s_1 s, \qquad t^{-1}s_1 t = s_2 s_1.$$

Since t^2 is commutative with every operator of G it results that

$$t^{-1}s_1 st = s_2 s_1^2 s = s, \qquad \text{or} \qquad s_2 s_1 = s_1^{-1} = t^{-1}s_1 t.$$

That is, *in every automorphism of order* 2 *some operator besides the identity must correspond to its inverse.* It also results that *the necessary and sufficient condition that an automorphism shall be of order* 2 *is that every commutator arising from the automorphism corresponds to its inverse under the automorphism.*

It is well known that automorphisms play a fundamental rôle in many problems in group theory, but little has been done toward a general theory of automorphisms. Even the automorphisms of order 2 present many difficulties at the present stage of development of group theory. Such a fundamental question as whether every group admits automorphisms of order 2 has not yet been answered. When G is an abelian group whose order exceeds 2 it clearly admits at least one automorphism of order 2.

The main results of the present paper relate to the case when G is abelian. In § 4 a few results are obtained which relate also to non-abelian groups, and the final theorem is an extension of results coming under the heading of the present paper.

---

* Presented to the Society (Chicago), April 9, 1909.

† FROBENIUS, Berliner Sitzungsberichte, 1895, p. 185.

§ 2. *Abelian groups of odd order.*

When G is an abelian group of odd order the identity is the only commutator arising from t and invariant under t. All the commutators which arise from transforming G by t constitute a group H, and to each operator of the quotient group $G/H = H'$ there corresponds one and only one operator of G which is commutative with t. The independent generators of G may be selected from H and these invariant operators under t. Hence the theorem: *The independent generators of an abelian group of odd order may be so chosen that in any given automorphism of order* 2 *each of these generators corresponds either to itself or to its inverse.* In particular, if an operator of order 2 transforms into itself an abelian group of odd order, it is possible to select the independent generators of this group in such a manner that each of them is transformed either into itself or into its inverse.

If a set of independent generators of G is so selected that the order of each of them is a power of a prime, every other possible set of such independent generators may be placed in a (1, 1) correspondence with this set. Hence it is easy to find the number of sets of conjugate operators of order 2 in the group of isomorphisms (I) of G. This may be accomplished as follows: Represent all the independent generators, which have been selected in the given manner, by letters in such a way that the generators of the same order are represented by the same letter and the generators of different orders by different letters. Form all the possible combinations of these letters, taking one, two, three, etc., at a time. The number of these combinations is the number of distinct sets of conjugate operators of order 2 in I. In particular, when G is of order p^α and of type $(1, 1, 1, \cdots)$ the number of complete sets of conjugate operators of order 2 in I is α.

By means of the results obtained in the preceding paragraph it is easy to determine the number of possible groups of order $2(2n+1)$ involving a given abelian subgroup of order $2n+1$. In fact, the number of these groups is exactly one more than the number of complete sets of conjugate operators of order 2 in the I of the abelian group of order $2n+1$. In particular, there are exactly $\alpha+1$ groups of order $2p^{\alpha m}$ involving the abelian group of order $p^{\alpha m}$ and of type $(m, m, m, \cdots)$. Since the group of isomorphisms of a Sylow subgroup of G involves just one invariant operator of order 2,* *the total number of invariant operators of order* 2 *in* I *is* $2^\lambda - 1$, λ *being equal to the number of distinct Sylow subgroups in* G.

For the purpose of determining the total number of operators of order 2 in I it is convenient to observe that the group of isomorphisms of any abelian group is the direct product of the groups of isomorphisms of its Sylow sub-

*Transactions of the American Mathematical Society, vol. 2 (1901), p. 260.

groups. If $m_1, m_2, m_3, \cdots$ represent the numbers of operators of order 2 in the groups of isomorphisms of the various Sylow subgroups of G, the number of operators of order 2 in I is $(m_1+1)(m_2+1)(m_3+1)\cdots-1$. Hence we may confine our attention to the determination of the number of operators of order 2 in the group of isomorphisms of an abelian group (P) of order p^m, p being an odd prime. When P is cyclic its group of isomorphisms involves only one operator of order 2, viz., the invariant operator of this order, corresponding to the isomorphism in which every operator of P corresponds to its inverse. In all other cases this group of isomorphisms involves non-invariant operators of order 2 and the number of distinct operators in a complete set of conjugates is evidently equal to the number of conjugates under I of the corresponding pair of subgroups H, H'.

In general, H, H' are any two independent subgroups of G whose product equals G. Hence the number of operators of order 2 in I is equal to the number of pairs of independent generating subgroups contained in G, when H, H' and H', H are regarded as distinct pairs, and H differs from the identity while H' may be the identity. This number is clearly equal to the sum of the quotients obtained by dividing successively the order of I by the products of the orders of the groups of isomorphisms of H, H' when H, H' represent successively all the possible types of pairs of independent subgroups (except that H is not the identity) contained in G. When H is the direct product of Sylow subgroups, the corresponding quotient is equal to unity and vice versa. Hence this is another way of seeing that the number of invariant operators of order 2 in I is equal to $2^\lambda - 1$, λ being the number of Sylow subgroups in G. Some of these results may be expressed in a convenient form as follows: *The number of operators of order 2 in the group of isomorphisms (I) of an abelian group (G) of odd order is equal to the sum of the quotients obtained by dividing successively the order of I by the products of the orders of the groups of isomorphisms of all the possible pairs of independent generating subgroups H, H' of G such that H is not of the same type in two pairs and that H is not the identity. Each of these separate quotients represents the number of operators of order 2 in a complete set of conjugates under I.*

To illustrate this theorem we shall employ it to determine the number of operators of order 2 in the well known group of isomorphisms of the abelian group of order $p^3 (p>2)$ and of type $(1,1,1)$. The order of I is evidently $p^3(p^3-1)(p^2-1)(p-1)$, and H, H' are of one of the following three types $(1), (1,1); (1,1), (1); (1,1,1), (0)$. The number of operators of order 2 in I is therefore equal to the sum of the following three quotients:

$$p^2(p^2+p+1),\ p^2(p^2+p+1),\ 1.$$

Hence this I involves two sets of $p^2(p^2+p+1)$ conjugate operators of order

2, and, in addition to this, it involves the operator of order 2 which transforms every operator of G into its inverse. It may be added that H, H' are necessarily products of Sylow groups whenever G is cyclic. This establishes contact between the given theorem and the well known result that the group of isomorphisms of a cyclic group is abelian. As the type of H' is completely determined by that of H, we may confine our attention to the latter in determining the number of sets of conjugate operators of order 2 under I.

§ 3. *Abelian groups of even order.*

When the order of G is the double of an odd number its group of isomorphisms is the same as if its order were this odd number. This is a special case of the theorem that the group of isomorphisms of G is the direct product of the groups of isomorphisms of its Sylow subgroup. Hence we may confine our attention to the case when the order of G is of the form 2^m. This case presents much greater difficulties than the one considered above. As every non-cyclic abelian group of order 2^m is the direct product of abelian groups whose invariants are equal to each other, we shall generally confine our attention to the determination of automorphisms of order 2 in such an abelian group. Throughout the rest of this section it will be assumed therefore that the order of G is 2^m and that all the invariants of G are equal unless the contrary is stated.

Representing the commutator subgroup of G by H, as was done in the preceding section, it is easy to prove that each of the independent generators of H must be of order 2, $2^{\alpha-1}$, or 2^α, when 2^α $(\alpha > 1)$ is the common order of the independent generators of G. In fact, since each operator of H is transformed into its inverse by t and since each operator of G is generated by an operator of order 2^α contained in G, an operator (s) of H, whose order exceeds 2, must be generated by an operator of order 2^α which generates only two operators that are commutative with t. This generator of s is transformed by t into itself multiplied by an operator of order $2^{\alpha-1}$, since $t^{-1}st = s^{-2}s$. As this operator of order $2^{\alpha-1}$ generates s^2, it results that *the square of every operator of H whose order exceeds 2 is generated by an operator of order $2^{\alpha-1}$ which is contained in H.* This proves that *the independent generators of H whose orders are divisible by $2^{\alpha-1}$ generate the squares of all the operators of H whose orders exceed* 2. Hence there are three cases to consider: 1) When all the operators of H except the identity are of order 2. 2) When H involves operators of order $2^{\alpha-1}$ but none of order 2^α. 3) When H involves operators of order 2^α. These three cases will be considered in the given order.

When H involves no operator of order 4, the order of H must divide 2^d, $d = m/\alpha$. The number of automorphisms of order 2, when the order of H is equal to 2^d, is equal to the order of the group of isomorphisms of H. In gen-

eral, when the order of H is 2^{d-l}, $l < d$, the number of these automorphisms is equal to the order of the group of isomorphisms of H multiplied by the product of the number of the subgroups of order 2^l in the group generated by all the operators of order 2 in G and the number of subgroups of order $2^{\alpha l}$ and of type $(\alpha, \alpha, \alpha, \cdots)$ contained in G. As the numbers of these subgroups are well known,* the problem of determining all the automorphisms of order 2 that arise in the given manner is completely solved. Hence the theorem: *If G is an abelian group of order $2^{d\alpha}$ and of type $(\alpha, \alpha, \alpha, \cdots)$, $\alpha > 1$, the number of automorphisms of order 2 which give rise only to commutators of order 2 is equal to the sum of the products obtained by multiplying the order of the group of isomorphisms of the abelian group of order 2^{d-l} and of type $(1, 1, 1, \cdots)$ by the product of the number of the subgroups of order 2^l and of type $(1, 1, 1, \cdots)$, and the number of subgroups of order $2^{\alpha l}$ and type $(\alpha, \alpha, \alpha, \cdots)$ contained in G, for every value of $l = 0, 1, \cdots, d-1$.*

When $\alpha = 1$, both G and H are of type $(1, 1, 1, \cdots)$ and hence each operator of H is commutative with t. From this it follows that the number of independent generators of H is equal to or less than half the number of these generators in G. It is also evident that the number of sets of conjugate operators of order 2 in I is equal to the number of possible orders of H; i. e., it is equal to the largest integer which does not exceed $d/2$. If the order of H is 2^λ the number of operators of H which are commutative with t is $2^{d-\lambda}$ and the number of operators of I which are conjugate with an operator corresponding to such an H is clearly equal to the product of the following three numbers: 1) The number of subgroups of order $2^{d-\lambda}$ in G. 2) The number of subgroups of order 2^λ in such a subgroup of order $2^{d-\lambda}$. 3) The order of the group of isomorphisms of H. The total number of operators of order 2 in the group of isomorphisms of the abelian group of order 2^d and of type $(1, 1, 1, \cdots)$ may therefore be found by addition of these products for every value of λ from 1 to the largest integer which does not exceed $d/2$.† For instance, the group of isomorphisms of the abelian group of order 2^4 and of type $(1, 1, 1, 1)$ contains two sets of conjugate operators of order 2 involving respectively 105 and 210 operators, as may easily be verified, since this group of isomorphisms is the alternating group of degree 8.

The simplest case in which H involves operators of order $2^{\alpha-1}$ is that in which every operator of G corresponds to its inverse. In this case it is evident that H is the direct product of d cyclic groups of order $2^{\alpha-1}$ and that the corresponding operator of order 2 in I is invariant. When $\alpha = 2$, this is included in the preceding two paragraphs. Hence we may assume $\alpha > 2$. Since the given automorphism corresponds to an invariant operator of order 2 in I, it

*Cf. Annals of Mathematics, ser. 2, vol. 6 (1904), p. 5.

†Bulletin of the American Mathematical Society, vol. 8 (1902), p. 391.

may be followed by each one of the group of automorphisms giving rise to commutators of order 2 only. If it is followed by any other automorphism G must involve invariant operators of order 4 under the automorphism, since each commutator arising from the automorphism corresponds to its inverse in the automorphism. Hence the theorem: *The number of the automorphisms of order* 2 *in which* H *involves all the operators of order* $2^{\alpha-1}$, $\alpha > 2$, *contained in* G *is one more than the number of automorphisms of* H *which are of order* 2 *and lead only to commutators of order* 2. As the latter number was considered above, the determination of these automorphisms is complete.

Suppose that H involves an operator (c_1) of order 2^α. We proceed to prove that G must contain an invariant operator of order 2^α in this case. In fact, if $t^{-1}s_1t = c_1s_1$ it is necessary that s_1 be of order 2^α and also independent of c_1. Hence the group generated by c_1 and s_1 is of order $2^{2\alpha}$. Since $t^{-1}c_1t = c_1^{-1} = c_1^{-2}c_1$, and $t^{-1}s_1^2t = c_1^2s_1^2$, it results that $c_1s_1^2$ is invariant under t, and it is evident that $c_1s_1^2$ is of order 2^α. This proves the theorem: *When* H *involves an operator of order* 2^α *an operator of order* 2^α *in* G *must correspond to itself in the automorphism of order* 2 *which gives rise to* H. It is also evident that $c_1s_1^2$ and c_1 generate groups which have two and only two common operators.

The group of order $2^{2\alpha}\{c_1, s_1\}$ cannot involve more than one commutator of order 2 since these commutators are invariant under t. It results from this that if H involves two independent generators $\{c_1, c_2\}$ of order 2^α, the group generated by c_2 has only the identity in common with $\{c_1, s\}$, as the latter involves no commutators under t except such as are generated by c_1. Hence t must transform a fourth independent generator s_2 as follows: $t^{-1}s_2t = c_2s_2$, whenever H involves at least two independent generators of order 2^α. This method of argument may be repeated until all the independent generators of order 2^α in H have been exhausted. As these arguments apply to any abelian group of even order the result may be stated as follows: *If the commutator subgroup arising from an automorphism of order* 2 *of any abelian group of even order* (K) *involves* n *independent generators of highest order in* K, *then* K *involves at least* $2n$ *independent generators of this order.*

It was observed in § 2 that every abelian group of odd order has an automorphism of order 2 giving rise to a commutator subgroup which involves all the independent generators of the abelian group, while the preceding theorem shows that the commutator subgroup arising from an automorphism of order 2 of an abelian group of even order cannot involve more than one half of the largest independent generators of the group. If this commutator subgroup involves exactly half of the independent generators of highest order, it cannot involve more than half of those of the second highest order whenever this order is also even, etc. All the operators of H which are not powers of operators of higher order contained in H must arise from operators of highest order in G.

§ 4. *Commutators of order p^α.*

In the preceding section it was observed that an automorphism of order 2 of an abelian group of even order must always give rise to at least one invariant commutator of order 2. Moreover if all the commutators arising from a given automorphism are of order 2 and correspond to themselves in the automorphism, this automorphism must be of order 2. In the present section we inquire into the properties of a non-abelian group (K) which has no commutator whose order exceeds 2 but which is not restricted in any other way. It is not assumed in this section that all the commutators of K arise from automorphisms of order 2.

Since all the commutators of K are of order 2, every operator (s) of order 2 in K must be transformed under K either into itself or into cs, where c is of order 2. Moreover, from the fact that both s and cs are of order 2 it results that c, s, and cs are three commutative operators which, together with the identity, constitute the four-group. That is, every complete set of conjugate operators of order 2 in K is composed of mutually commutative operators. In other words, all the operators of order 2 in K are found in invariant subgroups each of which is of type $(1, 1, 1, \cdots)$. From this it follows directly that K involves only one Sylow subgroup of order 2^m, and that the quotient group with respect to this Sylow subgroup is abelian. This result will be generalized in the following paragraph.

If the orders of all the commutators of a non-abelian group K are of the form p_1^λ, p_1 being a prime number, it follows directly that every Sylow subgroup of K whose order is not a power of p_1 is abelian. Moreover every operator of K that transforms into itself a subgroup of order p_α^γ, $(\alpha \neq 1)$, is commutative with every operator of this subgroup. Hence it results from a theorem due to FROBENIUS that K contains a characteristic subgroup of index p_α^δ, where p_α^δ is the highest power of p_α that divides the order of K.* As the orders of the commutators of this characteristic subgroup must also be of the form p_1^λ, we may repeat the given argument when the order of this subgroup is not itself of the form p_1^λ. Hence the following theorem: *If the orders of all the commutators of a non-abelian group K are of the form p_1^λ, p_1 being any prime number, K involves only one Sylow subgroup of order p_1^m and the corresponding quotient group is abelian.*

From the preceding paragraph it follows directly that every group in which the orders of all the commutators are powers of the same prime number must be solvable. It is also easy to see that K contains an abelian subgroup which is the direct product of Sylow subgroups of every order except p_1^m. This follows directly from the fact that all the Sylow subgroups of order p_α^δ are

* FROBENIUS, Berliner Sitzungsberichte, 1901, p. 1324.

contained in a characteristic subgroup of order $p_1^m p_a^\delta$. That such a group as K may involve more than one Sylow subgroup of every order except p_1^m, results directly from the fact that the metacyclic group of degree p and of order $p(p-1)$ is included among those which satisfy the conditions imposed on K.

URBANA, ILLINOIS.

RESOLUTION INTO INVOLUTORY SUBSTITUTIONS OF THE TRANSFORMATIONS OF A NON-SINGULAR BILINEAR FORM INTO ITSELF*

BY

DUNHAM JACKSON

In a recent number of the Transactions of the Connecticut Academy, E. B. WILSON obtains a necessary and sufficient condition that a linear transformation be factorable into two involutory transformations. If this condition is translated from the language of dyadics, which WILSON uses, into that of the algebra of matrices, its form suggests at once a theorem of FROBENIUS concerning transformations of a bilinear form into itself. A mere combination of these two theorems is sufficient to establish the following, which is simpler in statement than either:

I. *A necessary and sufficient condition that a linear transformation be such as to carry some non-singular bilinear form into itself, is that it be factorable into two involutory transformations.*

It will be seen that the part of this theorem relating to the necessity of the condition is a generalization of a theorem of P. F. SMITH† which WILSON uses in proving his theorem, and which is substantially as follows:

II. (SMITH). *A necessary condition that a linear transformation be such as to carry some non-singular quadratic form into itself, is that it be factorable into two involutory transformations.*

The converse of SMITH's theorem is not true, as is remarked by WILSON; a substitution which is factorable into two involutory transformations is not necessarily capable of carrying a non-singular *quadratic* form into itself. In fact, the question of the possibility of factoring a transformation into two involutory transformations has no essential relation to the subject of symmetric bilinear forms, as distinguished from bilinear forms in general.‡

* Presented to the Society (Princeton), September 13, 1909.

† P. F. SMITH, these Transactions, vol. 6 (1905), p. 13.

‡ For a necessary and sufficient condition that a transformation be capable of carrying a non-singular quadratic form into itself, see FROBENIUS, Crelle's Journal, vol. 84 (1878), p. 41. Compare with this the theorem numbered IV below.

The theorems of WILSON and FROBENIUS mentioned at the beginning of the article may be stated thus:

III. (WILSON).* *A necessary and sufficient condition that a linear transformation be factorable into two involutory transformations, is that each elementary divisor of its characteristic matrix be paired with another, whose degree is the same, and whose root is the reciprocal of the root of the first, except that an elementary divisor which is a power of* $(\lambda-1)$ *or* $(\lambda+1)$ *may be paired with itself.*

IV. (FROBENIUS.)† *A necessary and sufficient condition that a linear transformation be such as to carry some non-singular bilinear form into itself,‡ is that each elementary divisor of its characteristic matrix be paired with another, whose degree is the same, and whose root is the reciprocal of the root of the first, except that an elementary divisor which is a power of* $(\lambda-1)$ *or* $(\lambda+1)$ *may be paired with itself.*

Theorem I is an obvious consequence of III and IV. But while FROBENIUS in proving IV uses the method of the algebra of matrices, WILSON's proof of III depends on GIBBS's method of dyadics and double products, and also on the work of SMITH, who uses still another method. It is possible, however, to give a proof of WILSON's theorem depending only on the algebra of matrices, and so to obtain greater uniformity of treatment. For a part of the proof it is necessary only to rewrite WILSON's dyadics in the matrix notation; the use of SMITH's theorem in establishing the sufficiency of the condition is avoided by actually writing down the factors in all, as WILSON does in some, of the special cases that are first considered.

In the course of the demonstration, reference will be made to another theorem of FROBENIUS, slightly more general than the one already given:

V. (FROBENIUS.)§ *A necessary and sufficient condition that two linear transformations* α, β *be such as to carry some non-singular bilinear form into itself,‖ is that the elementary divisors of the characteristic matrix of* β *be obtainable from those of* α *by replacing each root of the characteristic equation by its reciprocal.*

From this, which is itself not difficult to prove, Theorem IV follows immediately. For if it is assumed that the two sets of variables in the bilinear form are to be subjected to the same transformation, then $\beta=\alpha'$, where α' denotes the conjugate of α, and the theorem just stated reduces to IV, since the elementary divisors of the characteristic matrix of α' are the same as those of α.

* E. B. WILSON, *Theory of Double Products and Strains in Hyperspace*, Transactions of the Connecticut Academy of Arts and Sciences, September, 1908; p. 41.

† Loc. cit., p. 34.

‡ That is, when applied to each of the two sets of cogredient variables.

§ Loc. cit., p. 31.

‖ That is, as FROBENIUS uses the terms, that a non-singular matrix ϕ exists such that $\alpha\phi\beta=\phi$.

To proceed with the proof of WILSON's theorem: Suppose at first, that α is the matrix of the product of two involutory transformations, so that

$$\alpha = \phi\psi \qquad \text{where} \qquad \phi^2 = I, \qquad \psi^2 = I.$$

Then

$$\alpha^{-1} = (\phi\psi)^{-1} = \psi^{-1}\phi^{-1} = \psi\phi,$$

since

$$\phi^{-1} = \phi, \qquad \psi^{-1} = \psi.$$

If β, γ are any two non-singular matrices, the elementary divisors of the characteristic matrices of $\beta\gamma$ and of $\gamma\beta$ are the same, since

$$\gamma\beta = \gamma(\beta\gamma)\gamma^{-1}$$

(see, for example, BÔCHER, *Introduction to Higher Algebra*, p. 286).

If γ is any non-singular matrix, the elementary divisors of the characteristic matrix of γ^{-1} can be obtained from those of γ by replacing each root by its reciprocal, by Theorem V above, for

$$\gamma\gamma\gamma^{-1} = \gamma.$$

In the present case, α, ϕ, ψ are necessarily non-singular, since $\phi^2 = \psi^2 = I$. Therefore the elementary divisors of the characteristic matrix of α^{-1} are, on the one hand, the same as those of α, and, on the other, the same as those of α with each root replaced by its reciprocal; and this amounts to saying that the condition stated in Theorem III is necessary.

Next, suppose that α is any matrix such that the elementary divisors of its characteristic matrix satisfy this condition. The form of the condition insures that α is non-singular. It is to be shown that α is the product of two square roots of the unit-matrix. It will be convenient to make use of the following

LEMMA. If α is a matrix expressible in the form $\phi\psi$, where $\phi^2 = \psi^2 = I$, then any matrix β, whose characteristic matrix has the same elementary divisors as that of α, is so expressible.

For β may be written in the form

$$\beta = \gamma\alpha\gamma^{-1} = \gamma\phi\psi\gamma^{-1} = \gamma\phi\gamma^{-1} \cdot \gamma\psi\gamma^{-1}$$

where

$$(\gamma\phi\gamma^{-1})^2 = \gamma\phi\gamma^{-1}\gamma\phi\gamma^{-1} = \gamma\phi^2\gamma^{-1} = \gamma\gamma^{-1} = I$$

and similarly

$$(\gamma\psi\gamma^{-1})^2 = I.$$

The truth of the following statements becomes obvious on writing down the formulæ:

If the n-rowed matrix β has elements different from zero only in an m-rowed principal sub-matrix $\bar{\beta}$, and the n-rowed matrix γ has elements different from zero only in the corresponding m-rowed sub-matrix $\bar{\gamma}$, then the product $\beta\gamma$ has elements different from zero only in the corresponding m-rowed principal sub-

matrix, and these elements are those of the matrix $\bar{\beta}\bar{\gamma}$ obtained by multiplying together $\bar{\beta}$ and $\bar{\gamma}$ as m-rowed matrices.

If the n-rowed matrices β, γ have elements different from zero only in the principal sub-matrices $\bar{\beta}, \bar{\gamma}$ (not necessarily of the same order), neither of which contains any row or column corresponding to a row or column of the other, then $\beta\gamma = 0$.

Therefore, if $\beta = \beta_1 + \beta_2 + \cdots + \beta_p$, $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_p$, where the non-vanishing elements of $\beta_1, \cdots, \beta_p$ form non-overlapping $m_1, \cdots, m_p$-rowed principal sub-matrices, and those of $\gamma_1, \cdots, \gamma_p$ occur in the corresponding sub-matrices, then

$$\beta\gamma = \beta_1\gamma_1 + \beta_2\gamma_2 + \cdots + \beta_p\gamma_p,$$

where in forming the products $\beta_i\gamma_i$, as far as the non-vanishing elements are concerned, the factors may be regarded as matrices either of order m_i or of order n.*

In particular, if $\beta = \beta_1 + \beta_2 + \cdots + \beta_p$ as before, and the sub-matrix in each β_i is factorable into two square roots $\phi_i\psi_i$ of the unit-matrix of order m_i, then β is factorable into two square roots of the unit-matrix of order n, for

$$(\phi_1 + \cdots + \phi_p)^2 = I, \qquad (\psi_1 + \cdots + \psi_p)^2 = I,$$

and

$$(\phi_1 + \cdots + \phi_p)(\psi_1 + \cdots + \psi_p) = \beta.$$

It will be shown next that the condition of the theorem is sufficient in the special cases that the characteristic matrix of the given matrix has just one elementary divisor $(\lambda \pm 1)^n$, or just two elementary divisors $(\lambda - a)^e$, $(\lambda - 1/a)^e$. By the lemma above, it will be enough to exhibit in each case a single matrix which can be factored into two square roots of the unit-matrix, and whose characteristic matrix has the elementary divisor or divisors in question. The work that has just been done will then make it easy to complete the proof of the theorem in general.

Let two matrices ϕ, ψ be defined as follows: The elements in the principal diagonal of each shall be alternately 1 and -1, and those just above the principal diagonal alternately 1 and 0, the 1's of ϕ, above the diagonal, correspond to the 0's of ψ; all the other elements are zero. If n, the order of the matrices, is 4, to take a definite case, the formulæ are

$$\phi = \begin{Vmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{Vmatrix}, \qquad \psi = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}.$$

* FROBENIUS, loc. cit., p. 18.

On multiplication, it appears that $\phi^2 = I$, $\psi^2 = I$, and $\phi\psi$ is a matrix whose characteristic determinant is equal to $(\lambda - 1)^n$; when $n = 4$, as above,

$$\phi\psi = \begin{Vmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

The first minor in the upper right-hand corner of the characteristic determinant does not vanish for $\lambda = 1$, so that there is just one elementary divisor $(\lambda - 1)^n$. Therefore any matrix whose characteristic matrix has only a single elementary divisor, of the form $(\lambda - 1)^n$, is factorable into two square roots of the unit-matrix. A similar result may be deduced for the case that there is a single elementary divisor $(\lambda + 1)^n$, by considering the product $(-\phi)\psi$.

Let the notation

$$\begin{Vmatrix} \alpha & \beta \\ \gamma & \delta \end{Vmatrix}$$

represent the matrix of order $2e$ having the e-rowed matrices α, β, γ, δ, in its corners, arranged as indicated. A similar notation might be employed for matrices of order ne. By actually following through the process of multiplication, it is fairly easy to see that the ordinary rule for matrix multiplication still holds, if the elements of the matrices are regarded, not as scalars, but as themselves representing matrices, all of the same order.*

Let $\bar{\phi}$, $\bar{\psi}$ be two square roots of the unit-matrix of order e, whose product $\bar{\phi}\bar{\psi}$ is such that its characteristic matrix has just one elementary divisor, $(\lambda - 1)^e$. It has been shown that such matrices exist. Let a be any number different from zero. Let

$$\phi = \begin{Vmatrix} 0 & \sqrt{a}\bar{\phi} \\ \frac{1}{\sqrt{a}}\bar{\phi} & 0 \end{Vmatrix}, \qquad \psi = \begin{Vmatrix} 0 & \frac{1}{\sqrt{a}}\bar{\psi} \\ \sqrt{a}\bar{\psi} & 0 \end{Vmatrix},$$

where $\sqrt{a}$ may denote either square root, provided it is given the same value throughout. Form the product of ϕ and ψ:

$$\phi\psi = \begin{Vmatrix} a\bar{\phi}\bar{\psi} & 0 \\ 0 & \frac{1}{a}\bar{\phi}\bar{\psi} \end{Vmatrix}.$$

* Regard each matrix as a sum of matrices, in each of which all the matrix-elements but one are zero.

The characteristic matrix of $a\bar{\phi}\bar{\psi}$ has just one elementary divisor $(\lambda - a)^e$, and that of $1/a\bar{\phi}\bar{\psi}$ has just one elementary divisor $(\lambda - 1/a)^e$. Therefore the characteristic matrix of $\phi\psi$ has just two elementary divisors, $(\lambda - a)^e$ and $(\lambda - 1/a)^e$. Furthermore,

$$\phi^2 = \psi^2 = \begin{Vmatrix} I_e & 0 \\ 0 & I_e \end{Vmatrix} = I,$$

where I_e represents the unit-matrix of order e, and I, that of order $2e$. Consequently any matrix whose characteristic matrix has just two elementary divisors $(\lambda - a)^e$ and $(\lambda - 1/a)^e$ is factorable into two square roots of the unit-matrix.

Return now to the matrix α, which was assumed to be such that the elementary divisors of its characteristic matrix satisfy the conditions of the theorem. It is possible to write down a matrix β, having these same elementary divisors in its characteristic matrix, and expressible in the form $\beta_1 + \beta_2 + \cdots + \beta_\rho$, where the non-vanishing elements of $\beta_1, \cdots, \beta_\rho$ form non-overlapping principal minors, and the characteristic matrix of the sub-matrix corresponding to each β_i has either just two elementary divisors, $(\lambda - a)^{e_i}$ and $(\lambda - 1/a)^{e_i}$, or just one elementary divisor, $(\lambda \pm 1)^{m_i}$. In either case, the sub-matrix is factorable into two square roots of the corresponding unit-matrix. Therefore β, regarded as a linear substitution, is the product of two involutory transformations, and hence the same is true of α.

HARVARD UNIVERSITY, CAMBRIDGE, MASS.,
July 1 1909.

ON SINGULAR POINTS IN THE APPROXIMATE DEVELOPMENT OF THE PERTURBATIVE FUNCTION*

BY

FRANK WALKER REED

§ 1. Introduction.

A method of computing approximately the coefficients of those terms of high order in the perturbative function which derive their importance from the presence in them of a small divisor has been set forth by POINCARÉ. As applications of the method to the problem of three bodies, POINCARÉ † has considered the case of coplanar orbits, one elliptic and one circular, and the case of coplanar elliptic orbits, both eccentricities being small. The latter case has been developed in detail by COCULESCO.‡ HAMY § has discussed the cases of an elliptic enclosing a circular orbit and a circular enclosing an elliptic orbit, the mutual inclination being small, and the case of circular orbits with any inclination. The latter of these was discussed anew by FÉRAUD || together with the case of an elliptic enclosing a coplanar circular orbit, and the case of a circular enclosing a coplanar elliptic orbit.

Owing to the high degree of the algebraic equations which define the singular points upon which the approximate values of the coefficients depend, it is found necessary, in the general case, to resort to numerical examples. This paper is devoted to a discussion of these algebraic equations in certain cases more general than those so far considered, and to an extension of the criterion of admissibility as employed by POINCARÉ. The singular points in the most general case and in an important particular case are found and classified. The results in the cases studied by POINCARÉ and COCULESCO are given first because of their usefulness in treating the more general problem. Before proceeding to

* Presented to the Society (Princeton) September 13, 1909.

† POINCARÉ. *Les Méthodes Nouvelles de la Mécanique Céleste*, vol. 1, Chap. VI.

‡ COCULESCO. *Sur les expressions approchées des termes d'ordre élevé dans le développement de la fonction perturbatrice*, Journal de Mathématiques, ser. 4, vol. 1 (1895).

§ HAMY. *Sur les inegalités d'ordre élevé*, Bulletin Astronomique, vol. 10 (1893). *Sur le développement approché de la fonction perturbatrice*, Journal de Mathématiques, ser. 4, vol. 10 (1894); ser. 5, vol. 2 (1896).

|| FÉRAUD. *Sur la valeur approché des coefficients d'ordre élevé dans le développement en series*, Annales de l'Observatoire de Bordeaux, vol. 7.

this discussion, it is necessary to set forth briefly the principles of POINCARÉ's method, but in doing so the general equations are derived at once.

It is required to develop a certain function F_1^0 in terms of the mean anomalies l and l' in the form

$$F_1^0 = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} A_{m_1 m_2} e^{i(m_1 l + m_2 l')}, \qquad i = \sqrt{-1}, \qquad \text{and} \qquad m_1, m_2 \text{ integers.}$$

The coefficients $A_{m_1 m_2}$ are given by the integral

$$A_{m_1 m_2} = \frac{1}{2i\pi} \int_{|z|=1} z^{-n-1} \Phi(z) dz,$$

where

$$\Phi(z) = \frac{1}{2i\pi} \int_{|t|=1} F(z, t) dt,$$

and

$$F(z, t) = F_1^0 t^{ad-bc-1} z^{-\frac{d}{c}}.$$

The new variables, t and z, have been introduced by the transformation

$$e^{il} = t^c, \qquad e^{il'} = t^{-a} z^{\frac{1}{c}},$$

and the new constants by the relations

$$m_1 = an + b, \qquad m_2 = cn + d,$$

where a, b, c, d are finite integers, small relative to n, and a and c are relatively prime. Now if $\Phi(z)$ is expanded as a Laurent series

$$\Phi(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} a_{-n} z^{-n}$$

and the result substituted in the expression defining $A_{m_1 m_2}$, we get

$$A_{m_1 m_2} = a_n.$$

A theorem by DARBOUX * can be employed to obtain approximate values of a_n, and consequently of $A_{m_1 m_2}$, for large values of n without computing the intervening coefficients.

The initial contour of integration, $|t| = 1$, may be deformed in any manner without affecting the value of the integral $\Phi(z)$ provided no singular point of $F(z, t)$, considered as a function of t with z as a parameter, is encountered. By varying z continuously the singular points of $F(z, t)$ are displaced continuously. Singular points of $\Phi(z)$ are those values of z for which two singular

* DARBOUX. *Sur l'approximation des fonctions de très-grands nombres*, Journal de Mathématiques, ser. 3, vol. 4 (1878).

points of $F(z, t)$, considered as a function of t, come into coincidence. If the points were initially, that is for $|z| = 1$, one inside and the other outside the contour $|t| = 1$, then the singular point z of $\Phi(z)$ is *admissible;* for, $\Phi(z)$ ceasing to be holomorphic, the theorem of DARBOUX is applicable.

The principal part of the perturbative function * is

$$F_1^0 = L^{-2} P'^{-\frac{1}{2}},$$

where

$$P' = (\xi - \beta\eta)(\xi_0 - \beta_0\eta_0) + \tfrac{1}{2}(\cos J - 1)[\gamma\xi\eta + \gamma_0\xi_0\eta_0 - (\beta_0\xi\eta_0 + \beta\xi_0\eta)],$$

$$\beta = L'^2 L^{-2} e^{i(\bar{\omega}' - \bar{\omega})}, \qquad \beta_0 = L'^2 L^{-2} e^{-i(\bar{\omega}' - \bar{\omega})},$$

$$\gamma = L'^2 L^{-2} e^{i(\bar{\omega}' + \bar{\omega})}, \qquad \gamma_0 = L'^2 L^{-2} e^{-i(\bar{\omega}' + \bar{\omega})}.$$

L^2, $\sin\phi$, l, u, $\bar{\omega}$ represent respectively the major axis, the eccentricity, the mean anomaly, the eccentric anomaly, and the longitude of perihelion of a planet B referred to the line of nodes and the center of gravity, D, of A and B in a system of three bodies A, B, C. L'^2, $\sin\phi'$, etc. are the elements of C referred to D and the lines of nodes. J is the inclination. ξ, η are defined by

$$\xi = \cos u - \sin\phi + i\cos\phi\sin u,$$

$$\eta = \cos u' - \sin\phi' + i\cos\phi'\sin u',$$

and ξ_0, η_0 are their conjugates. With this notation the coördinates of B and C respectively are given by the real and imaginary parts of

$$\xi L^2 e^{i\bar{\omega}}, \qquad \eta L'^2 e^{i\bar{\omega}'},$$

respectively.

The singular points of $F(z, t)$ with which we are concerned are the same as those of F_1^0. These are (a) the points for which u and u' as functions of l and l' cease to be uniform, given by

$$\frac{\partial l}{\partial u} = 1 - \sin\phi\cos u = 0, \qquad \frac{\partial l'}{\partial u'} = 1 - \sin\phi'\cos u' = 0,$$

and (b) the singularities of F_1^0, given by

$$P' = 0.$$

Making the substitutions

$$x = e^{iu}, \qquad y = e^{iu'}, \qquad j = \tfrac{1}{2}(\cos J - 1),$$

$$\tan\frac{\phi}{2} = \tau, \qquad 1 + \tau^2 = \sigma,$$

$$\tan\frac{\phi'}{2} = \tau', \qquad 1 + \tau'^2 = \sigma',$$

* This form of the perturbative function is due to FÉRAUD, loc. cit.

we get

$$\xi = \frac{(x-\tau)^2}{\sigma x}, \qquad \xi_0 = \frac{(1-x\tau)^2}{\sigma x},$$

$$\eta = \frac{(y-\tau')^2}{\sigma' y}, \qquad \eta_0 = \frac{(1-y\tau')^2}{\sigma' y}.$$

Then the equations defining the singularities become

$$(1) \qquad (x-\tau)(1-x\tau) = 0,$$

$$(2) \qquad (y-\tau')(1-y\tau') = 0,$$

$$P = (\sigma\sigma' xy)^2 P' = [\sigma' y(x-\tau)^2 - \beta\sigma x(y-\tau')^2]$$

$$(3) \qquad \times [\sigma' y(1-x\tau)^2 - \beta_0\sigma x(1-y\tau')^2] + j\sigma\sigma' xy[\gamma(x-\tau)^2(y-\tau')^2$$

$$+ \gamma_0(1-x\tau)^2(1-y\tau')^2 - \beta_0(x-\tau)^2(1-y\tau')^2 - \beta(1-x\tau)^2(y-\tau')^2] = 0.$$

Singular points of the *first species* of $\Phi(z)$ are given by the intersections of the curves (1), (2), (3). Singular points of the *second species* are those satisfying the condition for double roots of (1), (2), or (3), any one of which may be written in the general form in the original variables

$$f(z, t) = 0.$$

The condition for a multiple root of $f(z, t)$ is

$$f = 0, \qquad \frac{\partial f}{\partial t} = \frac{c\sigma x^2}{(x-\tau)(1-x\tau)}\frac{\partial f}{\partial x} - \frac{a\sigma' y^2}{(y-\tau')(1-y\tau')}\frac{\partial f}{\partial y} = 0.$$

Equations (1) and (2) will not give rise to points of the second species.

Consider equation (3), which may be written

$$P(x, y) = BC + j\sigma\sigma' xyA,$$

where

$$A = \gamma(x-\tau)^2(y-\tau')^2 + \gamma_0(1-x\tau)^2(1-y\tau')^2$$

$$- \beta_0(x-\tau)^2(1-y\tau')^2 - \beta(1-x\tau)(y-\tau')^2,$$

$$B = \sigma' y(x-\tau)^2 - \beta\sigma x(y-\tau')^2,$$

$$C = \sigma' y(1-x\tau)^2 - \beta_0\sigma x(1-y\tau')^2.$$

The equation $\partial P/\partial t = 0$ becomes, after reducing by means of $P = 0$,

$$R = B'C(x-\tau)(y-\tau') - BC'(1-x\tau)(1-y\tau')$$

$$(4) \qquad -jc\sigma x(y-\tau')(1-y\tau')\left[A - x\frac{\partial A}{\partial x}\right]$$

$$+ja\sigma' y(x-\tau)(1-x\tau)\left[A - y\frac{\partial A}{\partial y}\right] = 0,$$

where

$$B' = c(x+\tau)(1-y\tau') + a\beta(y+\tau')(1-x\tau),$$

$$C' = c(1+x\tau)(y-\tau') + a\beta_0(1+y\tau')(x-\tau).$$

§ 2. Coplanar Orbits.

The equations $P=0$ and $R=0$, which determine all the singular points of the second species in the most general case, admit of important simplifications in certain particular cases. If we suppose the inclination zero, then $j=0$, and we have

(1) $(x-\tau)(1-x\tau)=0$, (3) $B=0$, (5) $B'=0$,

(2) $(y-\tau')(1-y\tau')=0$, (4) $C=0$, (6) $C'=0$,

where (3), (4) are the factors of the reduced equation $P=0$, and (5), (6), those of $R=0$. In this case the singularities are as follows:

$B=0$, $x=\tau$, and $y=\tau'$ intersect in U, $C=0$, and $y=\tau'$ intersect in V and W',

$C=0$, $x=\frac{1}{\tau'}$ and $y=\frac{1}{\tau'}$ intersect in U', $B=0$, and $y=\frac{1}{\tau'}$ intersect in V' and W,

$C=0$, and $x=\tau$ intersect in E and Y', $x=\tau$, and $y=\frac{1}{\tau'}$ intersect in X,

$B=0$, and $x=\frac{1}{\tau}$ intersect in E' and Y, $x=\frac{1}{\tau}$ and $y=\tau'$ intersect in X',

$B=0$ and $C=0$ intersect in B', B, R, and R',

$B=0$ and $B'=0$ intersect in C', D, T, and F,

$C=0$ and $C'=0$ intersect in T', D', C, and F'.

These singular points have been discussed by Poincaré and Coculesco.

§ 3. The General Case.

The main difficulty which arises in the general case is that of solving the algebraic equations. Even when τ, τ', j are small, as is usually the case in nature, the approximations thus made possible do not so reduce the equations

that they admit of literal solutions. But the problem is not without interest from the theoretical standpoint, although the method it is necessary to employ is dependent upon numerical assumptions. In fact the results to be obtained in this manner have the same generality as those which would be furnished by a literal solution.

Since for $j = 0$ the equations $P = 0$ and $R = 0$ reduce to

$$B = 0, \qquad C = 0, \qquad B' = 0, \qquad C' = 0,$$

it is easily seen that all the singular points for a small inclination will lie near those given in § 2. The reducible curve $P = 0$ will lie near the curves $B = 0$ and $C = 0$, but $R = 0$, being irreducible, will not lie near $B' = 0$ or $C' = 0$.

Referring to equations (3), (4) of § 1, and recalling the fact that β, β_0 and also γ, γ_0 are conjugate complex quantities, we can easily verify that

$$A(x, y, i) = x^2 y^2 A\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{i}\right),$$

$$A - x\frac{\partial A}{\partial x} = A_1(x, y, i) = -x^2 y^2 A_1\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{i}\right),$$

$$A - y\frac{\partial A}{\partial y} = A_2(x, y, i) = -x^2 y^2 A_2\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{i}\right).$$

Substituting in P and R above $1/x, 1/y, 1/i$ for x, y, i, multiplying throughout by $x^4 y^4$, and changing signs in the latter function, we arrive at the identical expressions for P and R. Thus $P = 0$ and $R = 0$ are self-reciprocal equations in x, y, provided that β, β_0 and also γ, γ_0 interchange. This was to be expected, since the equations were originally in $\cos u$, $\cos u'$, $\sin u$, $\sin u'$ which involve x, y reciprocally. Consequently the discussion of the singular points will be very much abridged.

The equation $P = 0$ is an algebraic equation of the sixth degree; the term with j as a factor is also of the sixth degree. The highest power of x, and also of y, is the fourth. The curve has the axes as double asymptotes, and has also two oblique asymptotes. The equation $R = 0$ is of the seventh degree. As the highest power of x or of y is the fourth, the curve has three asymptotes parallel to each of the axes. Among these are the two axes. There is one oblique asymptote.

We come to consider certain restrictions which may be put upon the elements. In the planetary theory we are concerned usually with the case of orbits having small eccentricities. We proceed to apply this restriction to equations $P = 0$ and $R = 0$, and to note what simplifications take place. It was found in the simpler cases that the abscissas of half of the singularities were of the order τ or τ', and that the other half were of the order $1/\tau$ or $1/\tau'$. The same is true for

the ordinates of the singular points. If τ, τ' are sufficiently small we have approximately

$$\sigma = 1 + \tau^2 = 1, \qquad \sigma' = 1 + \tau'^2 = 1;$$

and if in addition x and y are small, we have approximately

$$1 - x\tau = 1, \qquad (1 - x\tau)^2 = 1, \qquad 1 - y\tau' = 1, \qquad (1 - y\tau')^2 = 1.$$

In the expression for A the coefficients of $(y - \tau')^2$ and $(1 - y\tau')^2$ may be written respectively

$$\gamma(x - \tau)^2 - \beta(1 - x\tau)^2, \qquad \gamma_0(1 - x\tau)^2 - \beta_0(x - \tau)^2,$$

which become, omitting terms of the second order, $-\beta$ and γ_0 respectively. Making in $P = 0$ the approximations indicated and arranging in powers of y, we get

$$P_1 \equiv y^3[-\beta x(1 + j)] + y^2[(x - \tau)^2 + \beta\beta_0 x^2 + 2\beta\tau'(1 + j)x]$$
$$+ y[-\beta_0 x(x - \tau)^2 - 2\beta\beta_0 x^2\tau' - \beta\tau'^2 x(1 + j) + j\gamma_0 x] + \beta\beta_0 x^2\tau'^2 = 0.$$

The equation $R = 0$ likewise takes a reduced form. We have approximately

$$A - x\frac{\partial A}{\partial x} = \gamma_0 - \beta(y - \tau')^2, \qquad A - y\frac{\partial A}{\partial y} = \gamma_0 + \beta(y^2 - \tau'^2).$$

Introducing these expressions in $R = 0$, simplifying as above, and arranging in powers of y, we get

$$R_1 \equiv y^3[a\beta'(x - \tau) + c\beta' x] + y^2[2c\tau(x - \tau) - 3c\beta'\tau' x]$$
$$+ y[-c\beta_0 x(x^2 - \tau^2) - a\beta_0(x - \tau)^3 - 2c\tau'\tau(x - \tau) - 2a\beta\beta_0\tau' x(x - \tau)]$$
$$- a\beta'\tau'^2(x - \tau) + 3c\beta'\tau'^2 x - jc\gamma_0 x + ja\gamma_0(x - \tau)$$
$$+ c\beta_0\tau' x(x^2 - \tau^2) + 2a\beta\beta_0\tau'^2 x(x - \tau) - c\beta'\tau'^3 x + jc\gamma_0\tau' x = 0,$$

where $\beta' = \beta(1 + j)$.

Thus we have a system of equations, (1), (2), $P_1 = 0$, $R_1 = 0$, for determining the singular points of $\Phi(z)$ which lie near the origin. It is unnecessary to discuss the geometry of the curves $P_1 = 0$, $R_1 = 0$. We know that in the region in which x, y are of the order τ, τ' these represent very closely the curves $P = 0$, $R = 0$.

The equations for P_1, R_1 are of the forms

$$P_1 \equiv \alpha_1^{(1)} y^3 + \alpha_0^{(2)} y^2 + \alpha_{(1)}^{(3)} + \alpha_2^{(2)} = 0,$$

$$R_1 \equiv \beta_0^{(1)} y^3 + \beta_0^{(1)} y^2 + \beta_0^{(3)} y + \beta_1^{(3)} = 0,$$

where the coefficients $\alpha_i^{(j)}$, $\beta_i^{(j)}$ are polynomials in x in which j is the highest and i the lowest power involved. Eliminating y from these equations we obtain the determinant

$$\Delta_1 \equiv \begin{vmatrix} \alpha_1^{(1)} & \alpha_0^{(2)} & \alpha_1^{(3)} & \alpha_2^{(2)} & 0 & 0 \\ 0 & \alpha_1^{(1)} & \alpha_0^{(2)} & \alpha_1^{(3)} & \alpha_2^{(2)} & 0 \\ 0 & 0 & \alpha_1^{(1)} & \alpha_0^{(2)} & \alpha_1^{(3)} & \alpha_2^{(2)} \\ \beta_0^{(1)} & \beta_0^{(1)} & \beta_0^{(3)} & \beta_1^{(3)} & 0 & 0 \\ 0 & \beta_0^{(1)} & \beta_0^{(1)} & \beta_0^{(3)} & \beta_1^{(3)} & 0 \\ 0 & 0 & \beta_0^{(1)} & \beta_0^{(1)} & \beta_0^{(3)} & \beta_1^{(3)} \end{vmatrix} = 0.$$

This equation is of the fourteenth degree in x, the lowest term being of the second degree. Thus $x = 0$ is a double root; the other twelve roots give the singular points D, F, T, C, B_1, B_2, R_1, R_2, U_1, U_2, U_3, U_4.

The equation $\Delta_1 = 0$ can be solved only when it is made numerical by giving particular values to the constants. The roots, which may be complex for general values of the coefficients, must first be separated and their values found by continued approximations. This procedure gives, indeed, the singular points required but leads to no criterion of admissibility. A more fruitful method is to trace the original curves $P_1 = 0$, $R_1 = 0$ for certain values of the elements, and then, varying these elements, to note what changes take place in the singular points both as to their positions and as to their admissibility.

The equation $R = 0$ is homogeneous in c, a; consequently a variation of the ratio c/a may be applied very easily. But in practice the initial choice of c/a will usually be sufficient. In fact, the important terms in the perturbative development are those corresponding to small values of

$$m_1 \nu + m_2 \nu',$$

where ν, ν' are the mean motions. We have approximately, for n very large.

$$a\nu + c\nu' = 0,$$

from which

$$\frac{c}{a} = -\frac{\nu}{\nu'} = -|\beta|^{\frac{3}{2}}.$$

The most important applications of the theory under discussion will be to the case of Jupiter and a small planet. The particular values of the elements are chosen in what follows with reference to this problem. Let L_1^2, $\sin\phi$, etc., be the elements as defined above of the orbit of one of the small planets and let L'^2, $\sin\phi'$, etc., be the elements of the orbit of Jupiter. Assume the value $\beta = 2$, then the successive reductions of $-a/c$ become $\frac{1}{2}$, $\frac{1}{3}$, $\frac{5}{14}$, etc. This choice

has the advantage of making all the singular points real in the case of zero inclination and coincident major axes, the necessary and sufficient condition being $c/a < 0$. For the purpose of more accuracy in representing the curves $P = 0$, $R = 0$ by means of $P_1 = 0$, $R_1 = 0$ we shall assume τ, τ', j somewhat smaller than in the case in nature just suggested. We assume accordingly that

$$a = -1, \quad c = 2, \quad |\beta| = 2, \quad \tau = .03, \quad \tau' = .01, \quad j = -.0005.$$

If the perihelia coincide at the node, we have

$$\beta = \beta_0 = \gamma = \gamma_0 = 2;$$

if at $\pi/2$ from the node,

$$\beta = \beta_0 = -\gamma = -\gamma_0 = 2.$$

On introducing these values, the equations $P_1 = 0$, $R_1 = 0$ become

$$P_1 = y^3[-2x] + y^2[5x^2 - .02x + .0009] + y[-2x^3 + .04x^2 - .0020x \mp .0010x] + .0004x^2 = 0.$$

$$R_1 = y^3[2x + .06] + y^2[-.0036] + y[-2x^3 - .10x^2 + .0068x \pm .0030x - .000024 \mp .000030] + .04x^3 - .0008x^2 - .000016x \mp .000020x = 0,$$

the double sign arising through $\gamma = \gamma_0 = \pm 2$.

In the table on the following page are the equations resulting from particular values of x near the origin. The roots of the equation are multiplied by 100 and the corresponding coefficients and roots are set down. In the first column the values of x are given; the two single and two double columns following give the coefficients of y^3, y^2, y and the constant term of the reduced equation. The three triple columns give the roots of the cubic in y. They represent respectively the roots for $j = 0$; $j = -.0005$, $\gamma = \gamma_0 = 2$; $j = -.0005$, $\gamma = \gamma_0 = -2$.

§ 4. The Criterion of Admissibility.

It has been stated in § 1 that for a singular point of $\Phi(z)$ to be admissible two singular points of $F(z, t)$ must have come into coincidence from opposite sides of the contour $|t| = 1$. But $F(z, t)$ is a multiform function of z, t of an infinite number of determinations. For $F(z, t)$ contains the expression P uniform in x, y, which are in turn multiform in z, t, as is seen from the relations

$$z = x^a e^{\frac{a\tau}{\sigma}\left(\frac{1}{x} - x\right)} y^c e^{\frac{c\tau'}{\sigma'}\left(\frac{1}{y} - y\right)}, \quad t = x^{\frac{1}{c}} e^{\frac{\tau}{c\sigma}\left(\frac{1}{x} - x\right)}.$$

The discussion for admissibility must then be made for each determination of $F(z, t)$, that is, upon each sheet of the corresponding Riemann surface.

$P_1 = 0$.

x	y^3	y^2	y		y^0		$j=0$			upper sign			lower sign		
0	0	+9	0	0	0		0	0	∞	0	0	∞	0	0	∞
.005	−1	+9¼	−9¼	∓5	+1		.13	1	7.9	.07	1 9	.1	i	i	8.5
.01	−2	+12	−18	∓10	+4		.27	2	3.8	.15	i	i	i	i	5.3
.02	−4	+25	−40	∓20	+16		.61	4	1.6	.30	i	i	i	i	5 5
.03	−6	+48	−78	∓30	+36		1.0	6	1.0	.40	i	i	i	i	7.0
.04	−8	+81	−144	∓40	+64		1.4	8	.70	.42	7.0	2.7	i	i	8.7
.06	−12	+177	−408	∓60	+144		2.3	12	.43	.35	11.	3.0	.57	1.7	12.
.10	−20	+489	−1800	∓100	+400		4.2	20	.24	.22	19.	4.5	.25	3.9	20.
−.005	1	11¼	11¼	±5	+1		−.10	−1	−9.9	i	i	−9.3	i	i	−11.
−.01	2	16	26	±10	+4		−.17	−2	−5.8	i	i	− .11	−.41	−.71	− 6.9
−.02	4	33	72	±20	+16		−.25	−4	−4.0	i	i	− .18	−.41	−1.5	− 6.3
−.03	6	60	150	±30	+36		−.27	−6	−3.7	i	i	− .21	−.37	−2.2	− 7.1
−.04	8	97	272	±40	+64		−.25	−8	−3.9	i	i	− .22	−.31	−2.8	− 9.0
−.06	12	201	696	±60	+144		−.23	−12	−4.5	−5.3	−11.	− .20	−.28	−3.9	− 1.3
−.10	20	529	2600	±100	+400		−.16	−20	−6.3	−6.6	−20.	− .15	−.17	−5.9	− 2.0

$R_1 = 0$.

0	6	−36	−24	∓30	0	0	0	−.6	6.6	0	−1.2	7.2	.0	.17	5.8
.005	7	−36	+7¼	∓15	−9½	∓10	i	i	4.9	i	i	5.4	− .02	.75	4.4
.01	8	−36	+32	0	−20	∓20	i	i	3.6	i	i	3.8	.0	1.2	3.3
.02	10	−36	+56	±30	−32	∓40	i	i	1.2	i	i	1.3	−.23	2.4	1.5
.03	12	−36	+36	±60	−12	∓60	1	1	1	i	i	1	1	−1.2	3.2
.04	14	−36	−40	±90	+64	∓80	−1.5	−3.0	1.0	i	i	.43	.95	−2.6	4.2
.06	18	−36	−408	±150	+480	∓120	−4.4	−5.3	1.1	4.2	−3.6	1.3	1.0	−5.2	6.
.10	26	−36	−2344	±270	+3040	∓200	−9.4	−9.5	1.3	8.9	−8.9	1.4	1.4	−10.	10.3
−.01	4	−36	−100	∓60	+4	±20	.04	−2.3	11	.15	−3.4	12.	i	i	10.
−.02	2	−36	−184	∓90	−32	±40	−.20	−4.0	22	.05	−5.8	23.	i	i	20.
−.03	0	−36	−264	∓120	−132	±60	−.53	−6.8	∞	−.20	−10.	∞	i	i	∞
−.04	−2	−36	−328	∓150	−320	±80	i	i	−1.1	i	i	−.52	i	i	−12.
−.06	−6	−36	−360	∓210	−1056	±120	i	i	−3.4	i	i	−1.8	i	i	− 6.7
−.10	−14	−36	+296	∓330	−4640	±200	i	i	−9 0	i	i	−7.7	i	i	−10.

For the particular case $\tau' = 0$ POINCARÉ has overcome the difficulty. Using this value of τ' in z above and solving for y, we get

$$y = z^{\frac{1}{c}} x^{-\frac{a}{c}} e^{-\frac{a\tau}{c\sigma}\left(\frac{1}{x} - x\right)},$$

in which it is seen that y is uniform in $z^{1/c}$, $x^{1/c}$, i. e., in z', x' if we put

$$x^{\frac{1}{c}} = x', \quad z^{\frac{1}{c}} = z'.$$

Also $P(x, y)$ is uniform in z', x', and in these variables $F(z, t)$ has only two determinations, namely those of the radical $P^{-\frac{1}{2}}$. This is effectively a reduction to one sheet, as these two determinations are equal in numerical value and opposite in sign. In the new variables the initial contour $|t| = 1$ becomes $|x'| = 1$. This method has been extended by COCULESCO to the case of small values of τ', where as a first approximation the eccentric and mean anomalies are considered equal.

In general it can be shown by expanding z' as a power series in τ', and neglecting terms above the nth power (as is permissible except for values of y of the same order as τ', such as occur along certain branches of the curve $P = 0$) that $F(z, t)$ becomes a function of multiplicity $2n$ in the variables z', x'. The geometrical meaning of this transformation as applied to the contour $|t| = 1$ is interesting, but the investigation is unnecessary to the present discussion. For it will be shown that the admissibility of points for the general case of $\tau' \neq 0$ can be determined from the discussion of the case of $\tau' = 0$.

In fact, POINCARÉ has shown that, in general, a singular point of $\Phi(z)$ cannot change its character as regards admissibility for any variation of the elements. The exceptional cases, as we shall see, are those for which there arises a change in the combinations of the singularities of $F(z, t)$ which form the singularities of $\Phi(z)$. We proceed to establish a theorem simpler in form than the original theorem by POINCARÉ, but embodying its essential point. This theorem may be enunciated thus:

I. *For the values* $|z| = 1$ *the singular points of* $F(z, t)$ *cannot cross the circle* $|t| = 1$.

Having seen how the circle $|t| = 1$ can be replaced by either $|x| = 1$ or $|x'| = 1$, we are at liberty to make the discussion in the plane of x. Consider the relation

$$z = x^a e^{\sigma}{}^{\left(\frac{1}{x} - x\right)} y^c e^{\sigma'}{}^{\left(\frac{1}{y} - y\right)}.$$

If we let z be a parameter, this relation is an implicit multiform function of the two complex variables x and y. We have seen that it has an infinite number of determinations. There corresponds then to the function $z(x, y)$ a Riemann

surface S with an infinite number of sheets. The surface varies when the parameter z is varied, and for $|z| = 1$ becomes the surface S_0. Upon this surface trace the circle C_0 defined by

$$|x| = |y| = 1.$$

This is possible, for if x be given a value of modulus unity then we can choose a value of y having the modulus unity in such a manner that z may have any value whatever of modulus unity. On the other hand if $|x| = |y| = 1$ and $x = y$, then there corresponds a unique point on the circle of unit radius in the z-plane.

Now to each point on the Riemann surface S there correspond definite values of x and y, and consequently definite values of the eccentric anomalies u and u'. The points on the circles

$$|x| = |y| = 1$$

give real values of the eccentric anomalies, i. e., a real point on each of the orbits. But in this case $F(z, t)$ has a singularity only when the distance between the two bodies is zero, that is, when

$$P = 0;$$

or, when the distance of one of the planets from the sun is zero, that is, when

$$1 - \sin \phi \cos u = 0,$$

or

$$1 - \sin \phi' \cos u' = 0.$$

Excluding then the case of intersecting orbits and also the limiting case of the problem of two bodies where the orbit reduces to a straight line, it is impossible that for any variation of the elements a point on the circle C_0 shall become a singular point of $F(z, t)$. Therefore a singular point of $F(z, t)$ can not cross the circle $|t| = 1$ for values $|z| = 1$.

A singular point of $\Phi(z)$ has been defined as being admissible when it is formed by the coming into coincidence of two singularities of $F(z, t)$ from the interior and exterior respectively of the initial contour $|t| = 1$. The theorem just proved may be applied to the question of the permanence in the character of admissibility of the singular points of $\Phi(z)$. To distinguish among these singular points certain definitions are useful. We assume that the elements of the orbits, with the exception of the anomalies, remain constant. Let A, B be two singular points of $\Phi(z)$ at which z' has the values z_1', z_2' and assume $|z_1'| < |z_2'| < 1$. When z' is made to vary starting from the point A, then the corresponding points in the plane of x' separate and move subject to the conditions defining singular points of $F(z, t)$. If in the plane of x' there is a

means of direct passage from A to B along a line of continuously increasing modulus of z, then B is *subordinate* to A.* The means of passage is the locus furnished by one of the equations (1), (2), $P = 0$.

If the point z' encounters no other singular points in its course, and if the points A and B are only double points, then A gives three terminal values in the plane of x', one proceeding from A directly and two proceeding from the branch separating at B.

We define a singular point as *primary* if it has no admissible subordinate point. It is *secondary* if it has an admissible subordinate point. A primary point may be either admissible or inadmissible; a secondary point is pseudo-admissible, for, although there correspond to it terminal points both within and without the circle $|x'| = 1$, there can be no change of elements which will bring it upon the circle of convergence $|z'| = r'$ without also causing it to pass through the circle of absolute value of its subordinate point, and the subordinate point itself determines the limit of convergence of the Laurent series.

With these definitions we proceed to the discussion of admissibility, taking up the different cases that may arise. It is requisite first to establish the theorem:

II. *A primary singular point of* $\Phi(z)$ *can not change its character as regards admissibility for any variation of the elements, and this character can be determined by varying* z' *along any path whatever of increasing modulus.*

Consider the method of formation of the singular points of $\Phi(z)$. The variables x and y satisfy certain equations of the series (1), (2), $P = 0$, $R = 0$. The intersections furnished by these equations give singular points of $\Phi(z)$ of the first and second species, and to such a point there corresponds a definite value of z', that is, a point in the plane of z'. Instead of finding the intersections indicated and then computing the values of z', it is simpler to introduce certain auxiliary curves to take the place of (1), (2), $P = 0$, $R = 0$, whose intersections give the required singular points of $\Phi(z)$. Let

$$\alpha(x, y) = 0$$

be any one of the curves (1), (2), $P = 0$, $R = 0$. We have

$$z'(x, y) = r'.$$

When r' varies in a given manner, the points of intersection of $\alpha = 0$ and $z' = r'$ describe certain curves; in particular, if z' describes the circle

$$|z'| = r',$$

the singular points arising from the intersections of the curves will cause x' to

* POINCARÉ, loc. cit., p. 308.

describe certain β-curves in its plane. The curves $\alpha = 0$ may be replaced by the corresponding β-curves. Mere intersections of the β-curves are not singular points of $\Phi(z)$; the tracing points must coincide for some value of z'.

For the initial circle $|z'| = 1$ the β-curves in the plane of x' do not intersect the circle $|x'| = 1$, by theorem I. Now let the circle $|z'| = r'$ contract, keeping the origin fixed. The β-curves sweep over certain portions of their plane. The significance of this is that corresponding to any continuous path whatever from any point on the outer circle $|z'| = 1$ to any point on the inner circle $|z'| = r'$, there are certain paths lying entirely within the region swept over by the β-curves. Suppose that when the circle $|z'| = r'$ is reached two tracing points of the β-curves coincide. Corresponding to this point in the x'-plane is a point on the circle $|z'| = r'$, namely one of the singular points of $\Phi(z)$. Now reverse the variation of z', and starting with the singular point let z' go to a value $|z'| = 1$ along any path whatever consistent with the continuous increase of the radius of the circle on which it is found. The points in the x'-plane accordingly trace definite loci in the swept-over area and rest finally on the initial β-curves corresponding to $|z'| = 1$. These curves being entirely within or entirely without the circle $|x'| = 1$, the terminal values of x' are definitely within or without the circle $|x'| = 1$ for any z'-path to the circle $|z'| = 1$; whence we conclude that the point can not change its character as regards admissibility for any change of the elements, for this would necessitate that some of the points of the β-curve should cross the contour

$$|x| = |y| = 1,$$

which we have shown to be impossible.

If in a series of singular points, connected by relations of subordination as defined previously, the one first encountered is inadmissible, the circle $|z'| = r'$ may be contracted over this singular point, since a deformation of the contour in the plane of x' is not rendered impossible. This may be repeated until an admissible point is reached. It is evident that the theorem applies to this point as well as to all the preceding points of the series, that is, to all the primary points of the problem.

Consider now all the points both primary and secondary in the z-plane connected in any manner whatever as required by the definition of subordination and as permitted by the method of varying z. If the elements be varied continuously these singular points likewise vary continuously. Their curves of connection must be deformed if necessary to correspond to the displacements. It is proposed to account for the behavior and the consequent character of the points for all possible changes on their relative positions.

Let A represent a typical singular point with B as an immediate subordinate

point, and consider any particular series of connected points. Let B pass through A. Then:

(1) If both were originally inadmissible they remain so after the change, for their combined terminations in the plane are still on the same side of the unit circle.

(2) If A were originally admissible and B inadmissible, the passage of B through A leaves A admissible and renders B admissible and secondary, i. e., pseudo-admissible.

(3) There is no case of B admissible and A inadmissible.

(4) If B is admissible, then A is pseudo-admissible, and the passage of B through A gives the following result:

(a) If A were intrinsically admissible, that is, if its admissibility did not depend upon a branch of B not common to A and B, then A remains admissible and B becomes pseudo-admissible.

(b) If A were not intrinsically admissible, then A ceases to be admissible, and B remains admissible.

This discussion covers all cases of interaction of primary singular points. This interaction produces certain secondary points. If we consider the passage of a secondary admissibility through an immediate subordinate primary admissibility, we shall have one of the above four cases, consequently the converse results hold. In fact, these converse relations are contained in the cases as already stated. We have: (1) is its own converse, (4) (a) is its own converse, (3) does not exist, (2) is the converse of (4) (b),

It has been assumed that B actually passed through A, but the results would be the same if it passed by it simply, i. e., passed through it in absolute value.

It should be remarked here that the variation of z from a secondary point to to $|z| = 1$ is not entirely arbitrary. For if all subordinate points are not taken account of, it may easily happen that we get an apparent inadmissibility when the point is actually admissible as found by the path through the subordinate point.

This completes the theory of admissibility. Before proceeding to the discussion of the points found in § 3 it should be remarked that there are for each point in addition $c - 1$ others of the same modulus and differing in amplitude by the arcs $k\pi/c$. POINCARÉ showed that these points are all admissible or inadmissible according as one of their number is admissible or inadmissible, and this holds true in the general case.

§ 5. Discussion of the Singular Points.

Having seen that the singular points which arise in case of particular elements can not, as a rule, change their character as regards admissibility for a change of elements, it is advantageous to make use of the simpler cases to determine

what points are admissible and to follow these points when certain of the elements are made to vary continuously, in order to determine in the general case the points that are admissible.

Accordingly we make direct use of the results of POINCARÉ and COCULESCO already cited. Let us designate three cases as follows: Case I, $j = 0$; Case II, $j \neq 0$, perihelia coincide at a node; Case III, $j \neq 0$, perihelia coincide at ninety degrees from a node. In the figures 1, 2, 3 the light and heavy lines are the curves in the case $j = 0$, $j \neq 0$ respectively, also the full lines are the curves $P_1 = 0$, $x = \tau$, $y = \tau'$, and the broken lines are curves $R_1 = 0$. The

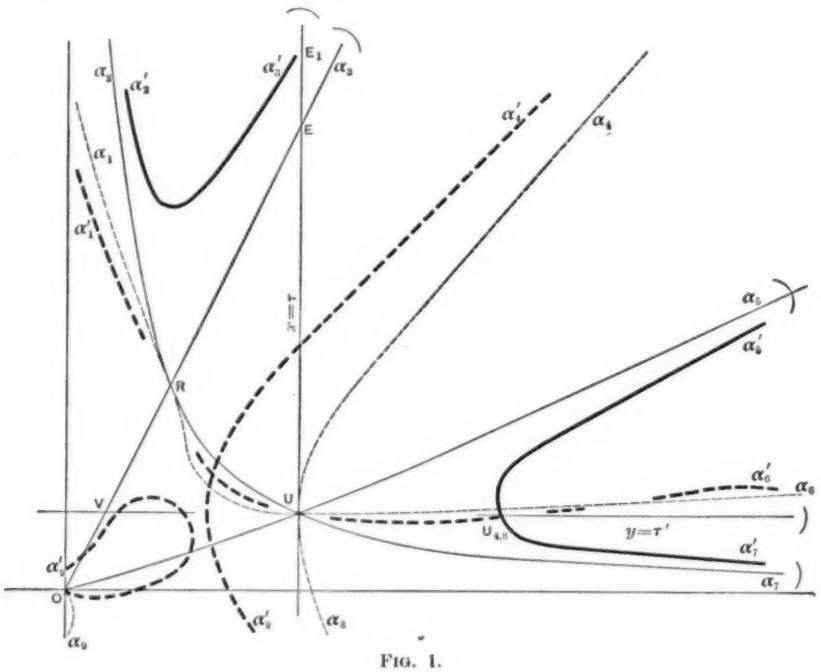

FIG. 1.

positions of the small arcs show where, with respect to the circle $|x'| = 1$, the points on the curves $P_1 = 0$, $x = \tau$, $y = \tau'$ lie for which $|z'| = 1$.

Case I. $j = 0$, $\bar{\omega} = \bar{\omega}'$. This is the case discussed by POINCARÉ and COCULESCO. The singular points have already been classified in § 2. Of the points of the first species B, R, E, W, X, Y are inadmissible and U, V are admissible. Of the points of the second species, C, F are inadmissible and D, T are admissible.

The variation of z is as follows: On the branches $\beta_4\alpha_3$, z decreases to C, which in this case coincides with B, is a minimum at C, increases from C to O,

changes at O from $|z| = \infty$ to $|z| = 0$, and increases from O through E. On the branches $\beta_2\beta_6$, $\alpha_2\alpha_7$, z decreases to D, is a minimum at D, increases through B to P, $|z| = \infty$ at P, $|z| = 0$ at Q, increases to F, which in this case coincides with R, is a maximum at F, decreases from F through V. On the branch $\beta_1\alpha_5$, z increases to T, is a maximum at T, decreases from T to O, changes at O from $|z| = 0$ to $|z| = \infty$, decreases from O to U, and increases from U. Along $x = \tau$ the value of z increases with y positive, and along $y = \tau'$ it increases with x positive.

Case II. $j \neq 0$, $\bar{\omega} = \bar{\omega}' = 0$. The curves in this case are given in Figures 1, 2, which are plotted with the values given in Table I. The branches α_1, α_2.

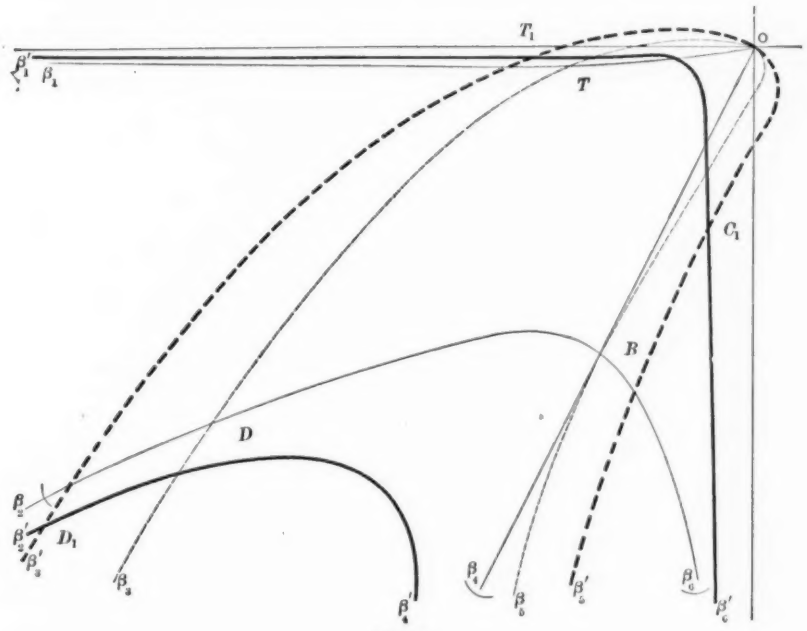

FIG. 2.

etc., are the curves of the Case I, just considered; the branches α'_1, α'_2, etc., are for present consideration. The curves $P_1 = 0$, $x = \tau$, $y = \tau'$ are represented in full lines, and the curves $R_1 = 0$ by dotted lines.

Consider the changes which the original curves undergo when the inclination is made to vary from zero, in other words when j varies from zero to a small negative quantity. The segments $\alpha_2 R$ and RO vary continuously to form the branch $\alpha'_2 O$. The segments $\alpha_3 R$, RU, $U\alpha_5$ form the branch $\alpha'_3 U_2 \alpha'_5$. The segments OU, $U\alpha_7$ form the branch $OU_1\alpha'_7$. The straight lines $x = \tau$, $y = \tau'$

are of course unchanged. The branches $\alpha_1\alpha_6$ and $\alpha_4\alpha_8$ become the branches $\alpha_1'\alpha_6'$ and $\alpha_4'\alpha_8'$ respectively.

The singular points for j sufficiently small will lie very near those of Case I. The point R is in reality a combination of what would be in general the points F and R. This coincidence is due to the choice of constants in the particular numerical case computed. A change of the ratio c/a is sufficient to separate the points F and R without changing the curve $P=0$ on which they lie. But these points can be separated otherwise. The branch $\alpha_1 RU$ is tangent to the branch $\alpha_2 RU$; this gives two coincident points of intersection at R. The branch $\alpha_1 RU$ also cuts the branch $\alpha_3 RO$ at R in a point of the second species R_2. Thus R is composed of the points F, R_1, R_2, all of the second species. When j varies in the manner indicated the point F remains real and takes a position F_1; the two points R_1, R_2 become conjugate imaginaries.

The point U in the simple case is composed of nine points. These arise as follows: each of the two branches of $R=0$ which pass through U cuts each of the branches of $P=0$ at the same point, giving rise to four points U_1, U_2, U_3, U_4 of the second species; each of the straight lines $x=\tau$, $y=\tau'$ cuts each of the branches of $P=0$ in the same point, giving rise to four coincident points of the first species, U_5, U_6, U_7, U_8; the point U is also the intersection of $x=\tau$, $y=\tau'$. Now vary j as before; the nine points all separate. U_1 and U_2 of the second species remain real. U_5 remains real. U_6 is imaginary in the figure, but it is evident that for sufficiently small values of j it is real. In fact it leaves U as a real point and different from the point E, these two points approaching each other along the line $x=\tau$. For a certain value of j the points E and U_6 coincide, and when j is given still larger values they become conjugate imaginaries. The points U_3, U_4 are conjugate imaginaries; the points U_7, U_8 are also conjugate imaginaries.

The remaining points V, of the figure, and W, Y, X, not represented in the figure, take positions near their respective original positions.

Discussion. — By virtue of the theorem established in § 4 it is not necessary to discuss the points E, F, R_1, R_2, V, W, X, Y, since we have already determined the nature of the corresponding points in the simpler case. We have only to consider the nine points corresponding to U.

The equation $P=0$ is of the form

$$P = ST + jA.$$

We proceed to find the form of this curve for the values of x lying between some value x of the order τ, and the value $x=1$. The product ST considered as a polynomial in y is of the fourth degree, while A is a polynomial of the third degree in y. By taking j sufficiently small we can make the curve

$$ST + jA = 0$$

lie as close as we choose to the curve

$$ST = 0$$

throughout the region under consideration; for the coefficient of the highest power in y remains unchanged, and all the coefficients of A are either finite or small.

Then for a given value of x the corresponding values of z' along similar branches in the two cases will be approximately equal. In the case of $j = 0$ the terminal values of x', when z' described the straight line of constant amplitude, were definitely within or without the circle $|x'| = 1$. It is possible in the present case to choose j small enough to make the terminal value of x', relative to the circle $|x'| = 1$, correspond exactly to those in Case I. The character of the real points U_1, U_2, U_5, U_6 is now evident.

U_6 is formed by the intersection of $x = \tau$ with $\alpha_3' U_2 \alpha_5'$. The two points on separation follow their respective branches in the direction of y increasing. They reunite in E. But E is inadmissible, therefore U_6 is inadmissible.

U_5 is inadmissible. The two points which separate are found to follow $U_5 O$ and $x = \tau$ in the direction of y decreasing. The terminal values of x are both without the circle $|x'| = 1$.

U_1 gives z' a minimum. When z' varies along the positive part of the real axis of its plane toward the value $z' = 1$, the two points x' on separation move along $U_1 O$ and $U_1 \alpha_7'$ respectively. The latter encounters in its course the point U_5. All the terminations for U_5 have been found to lie within $|x'| = 1$. The same is true for the terminations on the branch $U_1 \alpha_7'$, therefore U_1 is inadmissible.

U_2 is admissible. Since at this point z' is a minimum the values of x' on separation follow the branches $U_2 \alpha_3'$ and $U_2 \alpha_5'$. When z' attains the value unity we have for the first point $x' < 1$ and for the second $x' > 1$.

The points U_3, U_4, U_7, U_8 are left for consideration. These points are all imaginary but their amplitudes are very small when j is put small. Corresponding to each of these points the value of z' is imaginary with a small amplitude. Instead of varying z' along a line of constant amplitude, as was done in the case of the real points, let us vary z' along the straight line limited by the points z_0' and $z' = 1$, where z_0' is the value of z' at one of the singular points. The loci of the singular points which separate when z' is made to vary from the value z_0' are certain curves satisfying the relations

$$z(x, y) = \zeta, \qquad P(x, y) = 0,$$

for points of the second species U_3, U_4; or

$$z(x, y) = \zeta, \qquad y - \tau' = 0$$

for points of the second species, U_7, U_8, where ζ is a parameter varying along the straight line limited by z_0' and $z' = 1$. In the case of $j = 0$ this variation of ζ is along the real axis, and consequently the singular points on separation follow $P(x, y) = 0$ in the real plane. For j very small the singular points will describe curves lying not in the real plane, but very near the real part of $P = 0$. The approximation becomes closer as ζ approaches $\zeta = 1$, for which value the singular points have positions on the real part of $P = 0$ not far from those terminal positions found for $j = 0$.

Consider the singular points of the second species, U_3, U_4, formed by the intersections of the curves $P = 0$ and $R = 0$ in the neighborhood of the point U. Such an intersection implies that z' is a maximum or a minimum. In each case we know it to be a minimum. Then the four points x' after separating, two from U_3 and two from U_4, follow each of them one of the four segments of $P = 0$, which approach coincidence with UO, UR, $U\alpha_5$, $U\alpha_7$, respectively.

An examination of the order of interchange of the branches near U is necessary. But this discussion is practically obviated through *a priori* considerations. In fact, U represents an ordinary node of the function $P = 0$ in the case of $j = 0$. When j varies the interchange may take place in one of four different ways. The double point may continue to exist near U, or the two branches may cease to intersect and the continuous branches may lie near the original curve in any of three ways as follows: $\alpha_5'' UR$ and $\alpha_7'' UO$, $\alpha_5'' U\alpha_7''$ and RUO, $\alpha_5'' UO$ and $\alpha_7'' UR$. It can easily be shown that if the node continued to exist for variations of j, there would be one or three admissible singular points according as one of the two points of the second species is or is not on the segment $U_{10}\alpha_5''$. But it can be shown that for general values of j there is no node U_{10}. Consider the three other possibilities. By virtue of the continuity of the function P we see that each of the four segments which the singular points describe after separation is involved once and only once. We know that of the four terminal values of x' three are within the circle $|x'| = 1$ and one (on the branch α_5'') is without this circle. Thus of the two conjugate imaginary points, U_3, U_4, one is admissible and one is inadmissible. This distinction will be considered later.

The points U_7, U_8 which are the imaginary intersections of $y = \tau'$ and $P = 0$ in the neighborhood of U become real when $\bar{\omega} = \bar{\omega}' = \pi/2$, and are reserved for discussion under Case III.

The point U, which is the intersection of $x = \tau$ and $y = \tau'$, is or is not admissible according as one of the points U_7, U_8, supposed subordinate, is or is not admissible. This illustrates a distinction made in the foregoing section. In fact, if the points on separation were permitted to follow the line $y = \tau'$ only in the real plane, we should be led to conclude that U is necessarily inadmissible.

The curves as they exist in the third quadrant are represented by Fig. 2. The branches β_1, β_2, etc., are the curves of Case I, while β'_1, β'_2, etc., are the curves in the present case. From the figure it is seen that as j varies from zero, the line $\beta_1 TOB\beta_6$ becomes $\beta'_1\beta'_6$, and $\beta_2 DB\beta_4$ becomes $\beta'_2\beta'_4$. The branch $\beta_3 DTOB\beta_4$ becomes $\beta'_3 O\beta'_4$. At B three points coincide. These separate when j is made to vary, the point C remaining real, B_1 and B_2 becoming conjugate imaginaries. The points D, T, remain real and their abscissas increase in absolute value.

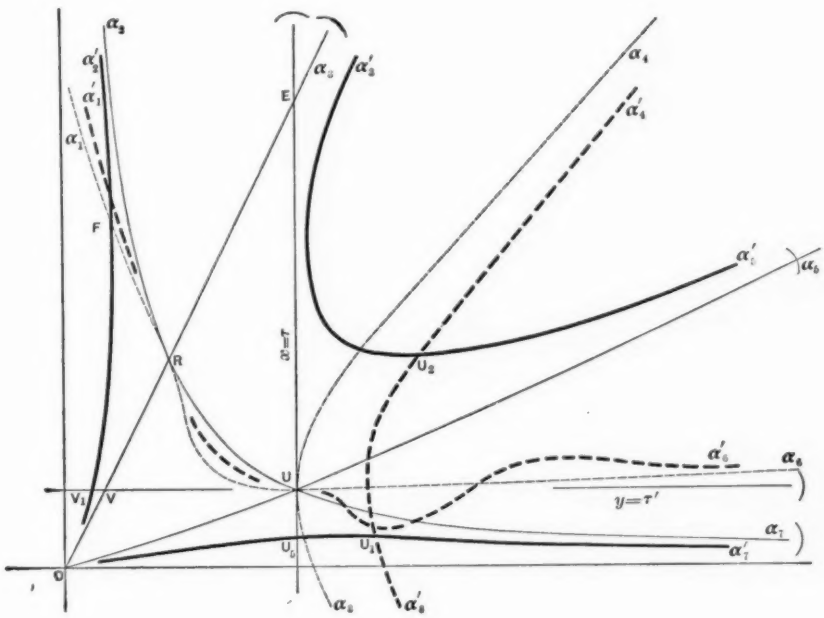

FIG. 3.

The points B_1, B_2, C are inadmissible and the points D, T are admissible, as was found in Case I.

Case III. $j \neq 0$, $\bar{\omega} = \bar{\omega}' = \pi/2$. In this case the curves as they exist in the first quadrant are represented by Fig. 3. They are plotted with the particular values given in Table I. The branches α_1, α_2, etc., are the curves in the case $j = 0$; the branches α'_1, α'_2, etc., are for present consideration.

When j varies from zero to the small value used in the table, the original curves vary as follows: $\alpha_2 R\alpha_3$ becomes $\alpha'_2\alpha'_3$; $ORUO$ remains a closed curve and recedes toward the point O; $\alpha_5 U\alpha_7$ becomes $\alpha'_5\alpha'_7$; the branches $\alpha_1 RU\alpha_6$

and $\alpha_4 U \alpha_8$ become respectively the branches $\alpha_1' R U \alpha_6'$ and $\alpha_4' \alpha_8'$; finally, the branch $\alpha_9 O$ passes through a loop form, then to the position $O\alpha_9'$. The order of the separation of the branches here is seen to be opposite to that of Case II.

The singular point at R breaks up into the points F, R_1, R_2. The points F, R_1 are conjugate imaginaries. R_2, the point lying below R, is real at first, but for the value of j employed in the figure it has become imaginary. When the loop formed by the segments $ORUO$ begins to recede toward the origin, the two points of intersection of this loop with a line RU, little different from the lower of the dotted lines RU, will be real and different. Designate by U_3 that one of the points near U. The value of j continuing its variation, the points U_3 and R_2 come into coincidence and then separate to become conjugate imaginaries. Consider the variation of z' on this loop when U_3 and R_2 are real. The loop is cut by the line $y = \tau'$ at the points V and U_7. The variation of z' is as follows: zero at O; OVR_2, increases; maximum, at R_2; R_2U_3, decreases; minimum at U_3; U_3U_7O, increases; infinity at O. Now let the loop decrease, the maximum point R_2 comes to coincide with the minimum point U_3, then z' continuously increases around the loop OV_1U_7O. The loop continuing to decrease, V_1 and U_7 approach each other, coincide, and then become conjugate imaginaries. The curve $\alpha_1' U \alpha_6'$ and the line $y = \tau'$ cut the curve $P = O$ at approximately the same point in the figure. The straight line $x = \tau$ cuts the curve $P = O$ in two imaginary points U_5, U_6.

Discussion. — By using the results of the two preceding cases the character of the singular points in the present case is easily determined.

F is inadmissible.

E is inadmissible.

R_1 has F and E as subordinate points and is itself inadmissible.

R_2 is a maximum point of z', and when z' increases the two points become conjugate imaginaries at first and then coincide at R_1. Thus R_2 is inadmissible.

U_4, the intersection of $\alpha_5' \alpha_7'$ and $U\alpha_6'$, is admissible. For, after separation, the two values of x' follow the branches α_5' and α_7' respectively, the terminal on the former branch being without and that on the latter branch being within the circle $|x'| = 1$.

U_8, subordinate to U_4 for sufficiently small values of j, is admissible. The tangent to the curve $U\alpha_6'$ at U is negative thus giving U_8 a position above U_4. The terminal value of x' in the directions of x increasing and α_5' are respectively within and without the circles $|x'| = 1$.

U has U_8 as a subordinate point and is consequently admissible.

U_7 is admissible; of its two branches one follows the curve U_7, O and the other passes through U and U_8 which are admissible.

U_3 is admissible; it has R_2 and U_7 as subordinate points, of which the latter is admissible.

V_1 is admissible; it has at first R_2 and U_7 as subordinate points, and later U_7 alone.

U_5 and U_7, the intersections of $x = \tau$ with $P = O$ are admissible and inadmissible respectively. For, of the four branches UR, UO, $U\alpha_7$, $U\alpha_5$, only the latter gives a terminal value of x' greater than unity. With the argument used for U_3 and U_4 of Case II, the conclusion is evident. The discussion for the third quadrant is not difficult and may be omitted.

§ 6. The Case of $\tau' = 0$, $j \neq 0$. Conclusion.

Before passing to conclusions relative to the general case, it is interesting to consider what simplifications arise on taking the eccentricity of the orbit of the second planet zero, the eccentricity of the orbit of the first being small, and the mutual inclination small.

Neglecting in the equation $P = 0$ the terms in τ' and arranging in powers of y, we have

$$y^2[\gamma\sigma jx(x-\tau)^2-\beta\sigma x(1-x\tau)^2(1+j)]+y[(x-\tau)^2(1-x\tau)^2+\beta\beta_0\sigma^2x^2] + \gamma_0\sigma jx(1-x\tau)^2-\beta_0\sigma x(x-\tau)^2(1+j)]=0.$$

An approximate expression for this curve in the neighborhood of the origin can be obtained in a very much reduced form by neglecting all but terms of relatively low order in the various coefficients. Thus considering x of the same order as τ, and neglecting all but terms of the second order in the coefficients, there results the equation

$$P_2 \equiv y^2[-\beta\sigma x(1+j)]+y[(x-\tau)^2+\beta\beta_0\sigma^2x^2]+\gamma_0\sigma jx=0.$$

Likewise the equation $R = 0$ becomes

$$R_2 \equiv y^2[a\beta(x-\tau)(1+j)+c\beta\sigma x(1+j)]+y[2c\tau(x-\tau)] + [-a\beta_0(x-\tau)^3-c\beta_0\sigma x(x^2-\tau^2)+j\{-c\gamma_0\sigma x+a\gamma_0(x-\tau)\}]=0.$$

The eliminant $\Delta_2 = 0$ of $P_2 = 0$ and $R_2 = 0$ may be found as in § 3. It has for its roots the abscissas of points corresponding to U, F, R_1, R_2, D, C, B_1, B_2.

Equations $P_2 = 0$, $R_2 = 0$ may be written in the expanded forms

$$P_2 \equiv y^2[s_{11}x]+y[s_{22}x^2+s_{21}x+s_{20}]+[s_{33}x^3+s_{32}x^2+s'_{31}x]=0,$$

$$R_2 \equiv y^2[t_{11}x+t_{10}]+y[t_{21}x+t_{20}]+[t_{33}x^3+t_{32}x^2+t'_{31}x+t'_{30}]=0,$$

where s'_{31}, t'_{31}, t'_{30} are so designated because they contain all the terms in j, and

to these terms the double sign is attached. Then Δ_2 becomes

$$[s_{33}x^3 + s_{32}x^2 + s'_{31}x][u_5x^5 + u^4x^4 + \cdots + u_0]$$
$$+ [t_{33}x^3 + t_{32}x^2 + t'_{31}x + t'_{30}][v_5x^5 + \cdots + v_0] = 0,$$

where the u, v are functions of the s, t as indicated. Or again,

$$\Delta_2 \equiv \lambda_8x^8 + \lambda_7x^7 + \cdots + \lambda_0 = 0.$$

Assuming the approximate values of the roots as found by plotting, they may be corrected by applying the well known formula

$$x = x_0 - \frac{\Delta_2(x_0)}{\Delta'_2(x_0)},$$

which is true for all simple roots, real or imaginary. The work may be reduced in the two cases under consideration by finding the four real roots first and then reducing the order of the equations.

In the two cases of $\bar{\omega} = \bar{\omega}' = 0$ and $\bar{\omega} = \bar{\omega}' = \pi/2$, the roots of Δ_2 may be calculated very rapidly by the above formula when j is varied little by little, since changes in j will affect s'_{31}, t'_{31}, t'_{30} and u_i, v_i in an additive manner simply.

Since the choice of β, τ can be made once for all, and since j can be made to assume any assigned value in the manner indicated, there remains for consideration the variation of the longitudes of the perihelia. Having taken $\tau' = 0$, we have $\bar{\omega} = \bar{\omega}'$ and

$$\beta = \beta_0 = \frac{L'^2}{L^2}, \qquad \gamma_0 = \frac{L'^2}{L^2}e^{-2i\bar{\omega}}.$$

When $\bar{\omega}$ is taken different from zero, imaginary terms affect the expressions λ_i. The roots of $\Delta_2 = 0$ become imaginary. They may be computed by the above formula as $\bar{\omega}$ varies from zero. The computation of x_0^n in the functions $\Delta_2(x_0)$, $\Delta'_2(x_0)$ is best effected by the use of trigonometric tables by writing

$$x_0^n = |x_0|^n e^{in\theta} = |x_0|^n(\cos n\theta + i \sin n\theta).$$

From the foregoing case the means of extension to the most general case is now evident. We have seen that it is necessary to take account of the analytic possibility that a singular point may become admissible or cease to be admissible, but slight attention to the geometry of the variation of the singular points as the elements vary will suffice to guard against error. Probably the selection of the primary admissible points, as well as their tracing in the x-plane, with the application of the principle of the limits of the roots of an equation, will answer in most of the practical applications. Approximations may usually be made in the manner employed here, and where necessary recourse may be had to the original equations of a higher degree.

Only points within the circle $|z| = 1$ have been considered in the foregoing discussion. After we have determined the coördinates x, y of the singular point which gives z nearest the circle, the point outside the circle $|x| = 1$ and nearest it is found as follows. We have

$$\Delta(x, i) \equiv X\Delta\left(\frac{1}{x}, -i\right) = 0,$$

where X is a power of x and Δ is the equation in x determining singular points. This shows that $1/x$ is also a root of Δ if $-i$ is put for i. But this is effectively a change of elements, namely an interchange of β with β_0 and of γ with γ_0. Make this change, then

$$\Delta(x, -i) \equiv X\Delta\left(\frac{1}{x}, i\right) = 0.$$

Thus we see that by putting $-i$ for i the knowledge of the inner point which conforms to the demands of the problem enables us to find the required outer point.

The simplest relations from which to begin the variation of the perihelia are

$$\bar{\omega} = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2},$$

with $\bar{\omega}$ coincident or opposite, which gives

$$\beta(=\beta_0) = \pm\gamma(=\gamma_0) = \pm\frac{L'^2}{L^2}.$$

The variation of c and a has not been considered. These constants enter the equation $R = 0$ only. This equation is homogeneous in c, a, so that if it is decided to vary the ratio c/a after a singular point is found, a factor may be entered readily in each term. Otherwise the problem may be begun anew for each choice of c/a, c and a being small integers.

THE UNIVERSITY OF ILLINOIS,
URBANA, ILL.

NOTES AND ERRATA

VOLUME 8

G. A. MILLER: *Generalization of the groups of genus zero.*

The theorems in italics on pages 12 and 13 should read as follows: *There are exactly six non-abelian groups whose two generators s_1, s_2 satisfy the equations $s_1^3 = s_2^5$, $(s_1 s_2)^2 = 1$. They are a group of order $60.32 = 1920$ and the direct products of the icosahedron group and the cyclic group of order 2^α, $\alpha = 0, 1, 2, 3, 4$.*

There are exactly six non-abelian groups whose two generators s_1, s_2 satisfy the equations $s_1^2 = s_2^3$, $(s_1 s_2)^5 = 1$. They are the icosahedron group, G_{120}, and the direct products of these groups and the cyclic groups of orders 5 and 25.

In the published theorem corresponding to the former of these two, the group of order 1920 was omitted owing to the incorrect conclusion that the order of t_2 could not be 160. The error which is corrected in the second of these theorems resulted from a mistake in the formula below the middle of page 12, which should read

$$(t_2^{-4} t_1 t_2^4 t_1^{-1})^n = t_2^2 (t_1 t_2^3)^n t_2^3 s_2^{-9n}.$$

In the published formula the last exponent was erroneously given as $-12n$. It should also be added that the theorems of this paper relate to non-abelian groups only, as the cases where the operators are commutative are so evident as to appear almost trivial. The restriction to non-abelian groups, however, should have been stated distinctly, but this was not done.

VOLUME 10

E. KASNER: *Natural families of trajectories.*

P. 202, l. 17.	*For*	FOURIER'S	*read*	FERMAT'S.
P. 214, l. 6 up.	"	the	"	a.